An overview of the quantum cognition research programme

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Abstract

The quantum cognition research programme concerns the application of quantum probability theory (the rules for how to assign probabilities to events from quantum mechanics, without any of the physics) to cognitive modelling. Quantum cognitive models have been applied to several areas of psychology and we provide a representative overview, with coverage in perception, memory, similarity, conceptual processes, causal inference, constructive influences in judgment, decision order effects, conjunction/disjunction fallacies in decision making, and other judgment phenomena. A challenge in this review is that the application of quantum theory in each area comes with its own unique challenges and assumptions. We present the empirical findings driving the application of quantum models for each area without theory, where this is possible, to allow an appreciation of the empirical drivers of model application; we discuss the quantum cognitive models in a way which allows consideration of their key features; we then cast a critical eye on the models and explore comparisons with non-quantum models. Our critical assessment of quantum modeling work is aimed at helping us answer questions such as is it worth persevering with quantum cognition models, what are the main weaknesses of such models, and what is their promise.

1. Introduction

In 2018 the quantum cognition research programme turned 23 years old (one can consider Aerts & Aerts, 1995, as the first major publication in this programme). It is a good time to consider questions such as what has the quantum cognition research programme achieved, what are its further prospects, and how does it relate to other influential research traditions in cognitive science.

We call quantum cognition the application of quantum probability theory to cognitive theory, where quantum probability theory refers to the rules for probabilistic inference from quantum mechanics, without any of the physics (for introductions suited to psychologists, in decreasing complexity, see Busemeyer & Bruza, 2011, Yearsley, in press, Pothos & Busemeyer, 2013; for introductory treatments for physicists see Hughes, 1989, Isham, 1989). In all the work we review, there is no assumption regarding physical quantum structure at the neurophysiological level: we assume a fully classical brain, such that neuronal processes can give to quantum-like structure at the macroscopic level. Thus, the quantum cognition research programme as defined here is non-overlapping with the controversial quantum brain hypothesis (Hameroff, 2007; Litt et al., 2006). One may wonder why the issue of neural implementation is challenging particularly for the quantum cognition research programme. For example, one would not start an overview of classical probability theory in cognition with a statement regarding neuronal application. The answer is that characteristic quantum effects, such as ontic uncertainty or spooky action at a distance, are often thought to require a quantum physical system, which would preclude relevance in a classical brain. However, we shall see that the application of quantum theory in psychology eschews such issues and, indeed, effects which may appear *weird* for physical systems may have familiar interpretations (such as contextuality).

Quantum probability theory is a set of rules for how to assign probabilities to events. Epistemically it is hardly different from the more influential classical/ Bayesian probability theory. The latter has led to a fertile research tradition, successful both in its descriptive coverage and a priori justification (e.g., Griffiths et al., 2010; Lake et al., 2015; Oaksford & Chater, 2007; Tenenbaum et al., 2011). However, classical probability theory is *just one* way to assign probabilities to events. Even outside psychological efforts to extend probability theory (e.g., Shafer, 1976; Tversky & Köhler, 1994), there are several systems for probabilistic assignment available to psychologists (potentially infinite, e.g., Sorkin, 1994). So, even if classical probability theory were uniformly successful in its application to cognitive theory, why restrict ourselves to the first reasonable solution for probabilistic inference we developed? An alternative probability theory might be sometimes more successful than classical probability.

In fact, classical probability theory has not gone unchallenged in cognitive modelling. Tversky and Kahneman have been most influential in pointing out discrepancies between the principles of classical probability theory and behavior, but many others have followed. The impact of Tversky and Kahneman's work is partly due to experimental tests that challenged the most basic principles of Bayesian inference and partly due to the persistence in the conflict between Bayesian prescription and intuition in their experiments – even when we are told what is the relevant Bayesian principle, it is sometimes difficult to overcome the (classically erroneous) intuition (cf. Gilboa, 2000). Inconsistencies between Bayesian principles and behavior are typically called fallacies and the most common theoretical route involves so-called heuristics and biases, that is, individual principles that can guide cognition, that are not part of a formal mathematical framework, but rather typically relate to other cognitive processes, such as attention, memory, or similarity. Such heuristics and biases have themselves had an extremely prominent place in psychological theory (Nobel prizes for Kahneman in 2002 and in 2017 for Thaler, both for economics; e.g., Kahneman et al., 1982; Tversky & Kahneman, 1973, 1974, 1983; Sloman, 1996).

The ongoing debate between classical probability explanations and ones based on heuristics and biases has been influential. Beyond purely descriptive arguments (which are not always straightforward to conclusively evaluate because of mimicries), a general epistemic reason for preferring theory based on heuristics and biases is that it often provides bridges between psychological processes, for example, decision making and similarity. A general epistemic reason for preferring models based on (classical) probability theory is that it provides a collection of coherent, interrelated principles and either all or none of them have to be involved in psychological theory. Note, even for psychological models based on classical probability theory, some non-probabilistic assumptions are needed, e.g., in terms of how to build representations from the available information. This is reasonable, unless there are so many such assumptions that the distinction between a formal classical probabilistic model and a heuristics/ biases one is blurred (Jones & Love, 2011).

Regarding quantum theory, it is worth noting that physicists had been initially extremely reluctant to adopt quantum theory – they were forced into it by empirical findings. Quantum probability theory works in physics because the structure of the theory matches the way the universe works, even if we don't understand why. In physics, quantum probability has led to several discoveries, which were previously unthinkable, partly because quantum probability theory embodies a way of thinking about probabilities different from the classical one. The motivation for considering quantum probability theory in cognition is analogous: our most basic point is that there have been several empirical findings in psychology which, superficially at least, indicate quantum structure. Such findings are often explained by heuristics, but the application of quantum theory can complement such explanations in terms of quantitative predictions and further generative value. The quantum cognition research programme is about exploring the potential of quantum probability in psychology, noting that it may be the case that certain behavioral findings may be outside the scope of both classical and quantum probability theories and may require models based on e.g. heuristics/biases.

A final preliminary question is why is a review of quantum cognitive models needed. There are two answers. First, the application of quantum theory in cognition is still relatively new, so it is important to pause, evaluate, and consider the merits of continuing with such applications. Second, there is an opportunity at this point to provide a reasonably comprehensive review.

2. What is quantum probability theory?

We begin with an informal presentation of quantum probability theory focused on its key properties and differences compared to classical probability theory. We do so by employing one of the most famous decision fallacies, Tversky and Kahneman's (1983) conjunction fallacy. In one of their conditions, participants were told of a hypothetical person, Linda (the arguably most famous hypothetical person in decision research), who was described in a way to make her look like a feminist, but not like a bank teller. Participants were asked to rank order a number of statements about Linda, according to how likely these were. The critical statements were that Linda is a feminist, Linda is a bank teller, and Linda is a feminist and bank teller. The results indicated that $Prob(feminist) > Prob(feminist \ \& \ bank \ teller) > Prob(bank \ teller)$. The conjunction fallacy is the finding that the conjunction is judged more probable than one of the marginals. Note, the conjunction fallacy has been extensively replicated and withstood the test of proposals for possible confounds or alternative explanations (notably conversational implicatures; e.g., Dulany & Hilton, 1991; Moro, 2009).

The conjunction fallacy is problematic for classical probability theory because the probabilistic model in classical theory is essentially volumetric, which means that the universe of possibilities can be thought of as a generalized volume and more specific questions/ possibilities are subsets of this volume. Thus, a more complex possibility (the conjunction between two predicates) can never be more probable a corresponding simpler one (either of the two predicates; Figure 1). The classical impossibility of the conjunction fallacy immediately becomes apparent if we recast the problem with countable instances, for example, compare number of days on which it rained and snowed in London in 2017 compared to the number of days on which it just rained. In Tversky and Kahneman's (1983) seminal demonstration, we have classical probabilities as subjective degrees of belief, but classical probabilities based on frequencies and classical probabilities based on subjective beliefs are equivalent (de Finetti et al., 1993).

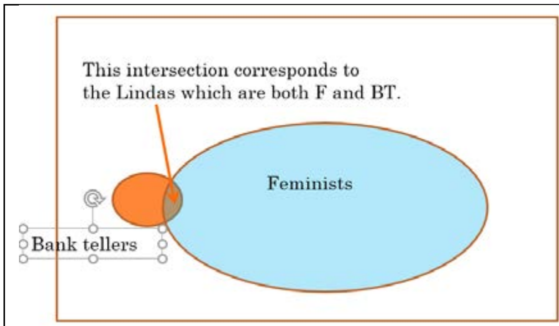

Figure 1. Consider a sample space of all possible Linda's consistent with the story. The ones for which both properties of bank teller and feminist are true can never be more than those for either individual possibility.

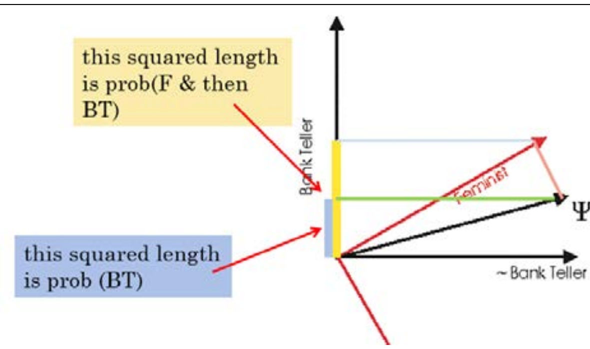

Figure 2. A caricature of Busemeyer et al.'s (2011) model for the conjunction fallacy, based on one-dimensional subspaces. ψ is the mental state and *F*, *BT* the feminist, bank teller properties.

Quantum probability theory is based on subspaces of a large space representing the universe of possibilities. In that universe we can have subspaces of varying dimensionalities corresponding to different possibilities/ questions. The relation between subspaces typically needs to be specified and any probabilistic computation depends on the state of the system under consideration. Psychologically the system typically corresponds to the mindset of the participant prior to a probabilistic inference. Classically, the mindset of participants is typically not explicitly specified, though such information can be introduced through conditionalization.

Figure 2 shows a caricature of the quantum model for the conjunction fallacy (Busemeyer et al., 2011). Several simplifications are made: first, one-dimensional subspaces are employed to represent the possibilities that Linda is a bank teller or a feminist. For the question of whether Linda is a bank teller, we have a so-called basis vector for the possibility that she is indeed a bank teller and another one for the opposite possibility. The basis vectors corresponding to all outcomes of a question are called a basis set. Second, the overall space is a two-dimensional real vector space, instead of an *n*-dimensional complex space. Third, the mindset of participants prior to a decision is represented by a state vector, instead of a density matrix which is the more general approach. When the state of the system is represented as a state vector, then probabilistic computation involves computing the squared length of the projection of the state vector onto the relevant subspace. For example, if we are interested in how probable possibility A is, when the system is represented as $|\psi\rangle$[1] (or just ψ), then we have to compute $|P_A|\psi\rangle|^2$, where P_A is linear operator computing the projection of a vector to subspace A (it is often called a projector, projection operator, or measurement operator), $|\psi\rangle$ is a normalized column vector, and $|P_A|\psi\rangle|^2$ expressed as matrix operations gives $\psi^\dagger P_A \psi$, where $\psi^\dagger$ is the conjugate transpose of ψ. Other technical details are that these vector operations take place in a Hilbert space, which is a complex vector space with some additional properties, and that if the system were represented as a density matrix ρ, probability for A would be given as $Trace(P_A \rho)$, where the trace operation sums diagonal elements of a matrix.

The Figure 2 representation shows that the initial mindset of participants after reading the Linda story is such that they are likely to consider Linda a feminist and not a bank teller, consistently with Tversky and Kahneman's (1983) design. The relation between the feminist and bank teller rays is determined by imagining a feminist Linda (so placing the state vector along the feminist ray) and considering whether for such a person we want small or large projections (probabilities) to the bank teller ray. The assumption made in Figure 2 is that a feminist Linda is neither particularly likely nor particular unlikely to be a bank teller.

This brings us to the first unique property of quantum probability theory (here and elsewhere) relative to classical probability theory. Questions/ possibilities in quantum theory can be incompatible, so that certainty about one introduces unavoidable uncertainty about the other and vice versa (in Figure 2 a state vector contained within the bank teller subspace has non-zero projections to both the feminist and non-feminist possibilities). Incompatible events in quantum theory give rise to uncertainty relations. In physics such uncertainty relations can imply, for example, that it is impossible to concurrently know the position and momentum of a particle and has led to extensive debate (for an introductory discussion see Isham, 1989). In psychology, a basic way to approach uncertainty relations is that accepting one possibility (e.g., that Linda is a feminist) creates a unique perspective or mindset for other, incompatible ones (e.g., that Linda is a bank teller). For incompatible possibilities, a joint probability distribution does not exist (since we cannot be certain for all combinations). In classical probability theory, possibilities can only be compatible, for which we must always specify a complete joint probability distribution. Compatible possibilities exist in quantum probability theory too, but for such possibilities quantum and classical predictions can converge.

Because uncertainty relations require the non-existence of joint probability distributions, conjunctions involving incompatible questions have to be assessed through a sequential projection operation, for example, $Prob(F \; \& \; then \; BT) = |P_{BT} P_F |\psi\rangle|^2$, where P_{BT} and P_F are projectors to the

[1] This is the so-called Dirac's Bracket notation, because $\langle a|$ is a bra (a row vector), $|b\rangle$ (a column vector) and together they make a bra-ket, $\langle a|b\rangle$, which is the dot product between vectors *a*, *b*.

corresponding possibilities; they are non-commuting, i.e., $P_{BT}P_F \neq P_F P_{BT}$, as the *F*, *BT* questions are assumed incompatible. In Figure 2, the direct projection to the BT subspace is shorter (lower probability) than the projection to the F subspace first and then to the BT one, corresponding to $P_{BT}P_F|\psi\rangle$. Though this caricature illustration is quite simple, it still provides an existence demonstration for how $|P_{BT}P_F|\psi\rangle|^2 > |P_{BT}|\psi\rangle|^2$ according to quantum probability theory. One question is whether the probability of F and then BT in quantum theory are equivalent to a conditional probability BT given F in classical theory. This is not the case, since in quantum theory $|P_{BT}P_F|\psi\rangle|^2 = |P_{BT}|\psi_F\rangle|^2|P_F|\psi\rangle|^2 \equiv Prob(BT|F)Prob(F)$, so that the notion of conditional probability in quantum theory (embodied in the Luder's law, e.g., Hughes, 1988) is distinct from sequential conjunction. The reason why sequential conjunction cannot be reduced to (some notion) of classical conditional probability is uncertainty relations, that require re-introduction of uncertainty with every projection.

Uncertainty relations lead to interference effects. For example,

$$\begin{aligned}|P_{BT}|\psi\rangle|^2 = |P_{BT}I|\psi\rangle|^2 = |P_{BT}(P_F + P_{\sim F})|\psi\rangle|^2 = |(P_{BT}P_F + P_{BT}P_{\sim F})|\psi\rangle|^2 \\ = |P_{BT}P_F|\psi\rangle|^2 + |P_{BT}P_{\sim F}|\psi\rangle|^2 + |P_F P_{BT}P_{\sim F}|\psi\rangle|^2 + |P_{\sim F}P_{BT}P_F|\psi\rangle|^2 \\ = Prob(F \;\&\; then\; BT) + Prob(\sim F \;\&\; then\; BT) + \Delta\end{aligned}$$

So, we end up with an expression which appears like the quantum analogue of the law of total probability, plus Δ. Δ is an interference term, which can be positive or negative, and allows quantum probabilities to violate the law of total probability.

Another key characteristic of quantum probability theory concerns the kind of uncertainty reflected in a state such as $|\psi\rangle = a|BT\rangle + b|{\sim}BT\rangle$, which is called a superposition (*a*, *b* are called amplitudes and are complex numbers whose squared moduli are probabilities). Such uncertainty is ontic, so that prior to a measurement (decision) there is no reality regarding whether Linda is BT or ~BT, and a measurement creates a possibility into being (cf. Atmanspacher & Primas, 2003). For example, post measurement the state might be $|\psi\rangle = |BT\rangle$. Thus, resolving a question changes the state of the system and this process is called the collapse of the state vector. In quantum theory some states can incorporate epistemic uncertainty too, which reflects lack of knowledge. By contrast, classical probability theory only involves epistemic uncertainty (for a more subtle expression of the difference between classical and quantum uncertainty see Griffiths, 2013, and Spekkens, 2007). In physics, the meaning of superpositions and the collapse of the state vector has led to extensive debate. For example, what does it mean for an electron to not have a position prior to measurement? In psychology, if anything, the situation is the converse, since it is often reasonable to assume that decisions change the mental state.

The state of a quantum system can change through time evolution. We distinguish between two situations, when a system is isolated from and when it interacts with its environment. Psychologically, this can be related to whether a task is solved without or with influence from the experiences or general knowledge of a person. Unitary dynamics and Schrodinger's equation concern isolated systems; open-systems dynamics and Lindblad's equation when there is interaction with the environment. A key property of unitary evolution is that probabilities keep oscillating, that is, there is always change with time. By contrast, with open-systems dynamics probabilities eventually stabilize to some pattern. For both kind of dynamics, the Hamiltonian operator contains information for how probabilities change with time.

Finally, superposition states can be compositional or non-compositional. Consider a participant deciding not just whether Linda is a BT, but whether Jane is a BT too. Suppose that the judgment for Linda is independent from the judgment for Jane. Then we can write

$$\begin{aligned}|\psi\rangle = ab|BT_{Linda}\rangle|BT_{Jane}\rangle + ab'|BT_{Linda}\rangle|{\sim}BT_{Jane}\rangle + a'b|{\sim}BT_{Linda}\rangle|BT_{Jane}\rangle \\ + a'b'|{\sim}BT_{Linda}\rangle|{\sim}BT_{Jane}\rangle \\ = (a|BT_{Linda}\rangle + a'|{\sim}BT_{Linda}\rangle)\otimes(b|BT_{Jane}\rangle + b'|{\sim}BT_{Jane}\rangle)\end{aligned}$$

That is, the state for whether Linda, Jane are bank tellers is given as the tensor product for the state for each person individually. Alternatively, suppose we know Jane is a very good friend of Linda's – so that we think that Linda, Jane would be very similar. Then, an appropriate mental state would be $|\psi\rangle = x|BT_{Linda}\rangle|BT_{Jane}\rangle + y|{\sim}BT_{Linda}\rangle|{\sim}BT_{Jane}\rangle$. Such a state is called a Bell state and it illustrates the key quantum property of entanglement. Here, a decision for either Linda or Jane forces an outcome for the other. Quantum entanglement precludes the specification of a joint probability distribution for all possible question combinations, that can factorize into the probability distributions for each component question. Put differently, entanglement means that, for component questions A and B, we cannot write the state for the combined question as $|A\rangle\otimes|B\rangle$, that is the combined state cannot be constructed by independently combining parts of the representation from $|A\rangle$ and then $|B\rangle$. In physics, entanglement has led to endless debate, because the two systems can be spatially separated. However, when dealing with mental states, entanglement is less philosophically challenging (it can be attributed to some connectedness between the systems, e.g., the different component concepts of a composite one).

Entanglement can allow supercorrelations. Consider two systems A and B, and two binary questions for each system, A1, A2 and B1, B2. We can compute expectation values (labelling the binary outcomes for each question as ± 1) for all pairs composed of one question from system A and one question for system B. An ingenious result by Bell shows that classically these expectation values have to be bounded as follows $|E[A1, B1] + E[A1, B2] + E[A2, B1] - E[A2, B2]| \leq 2$ (this is the CHSH form of the Bell inequality, Clauser et al., 1969; Bell, 2004). Here, classical means that the two systems can be perfectly correlated or anticorrelated, but that the two systems do not interact when measured. Bell showed that if A, B are described with a quantum entangled system there are pairs of questions for which the Bell bound is exceeded (and another bound applies, the so-called Tsirelson bound). In this sense, entangled quantum systems supercorrelate.

Overall, the key features of quantum theory that lend themselves to psychological application are primarily interference effects, the collapse of the state vector, entanglement, and supercorrelations, though of course these features are closely inter-dependent. Physicists have sought to reduce all of quantum theory into a single fundamental feature (e.g., superposition; Harding 2001), but such considerations are beyond the present scope. Table 1 summarizes the main terms.

Quantum term	Meaning	Additional note/ example
Hilbert space	The representation space in quantum theory.	Analogous to multidimensional spaces in standard cognitive theory.
State vector	This contains all information about the system of interest.	For example, the mental state of participants.
	The transition from uncertainty, before a decision, to certainty is called the collapse of the state vector.	When a response is made, the state vector has to identify with the outcome of the response.
	The state vector is typically a superposition for particular questions. Superpositions encode ontic uncertainty.	If you have a superposition for the position of an electron, then its position does not exist prior to a measurement.
Density matrix	A more sophisticated way to represent the state of the system, including classical uncertainty about the system state.	
Incompatibility	A joint probability distribution for incompatible questions does not exist and certainty for one introduces uncertainty for	In psychology, this is typically interpreted as contextuality or perspective dependence.
Uncertainty relations, interference effects	When dealing with incompatible questions, there are typically uncertainty relations and the law of total probability does not hold, because of interference effects.	In Figure 2, interfernece effects allow the conjunction fallacy.
Subspaces	Each question outcome is represented by a subspace.	For example, in Figure 2, Linda being a bank teller is a subspace.
	To compute the probability for a question outcome, we have to project the state vector to this subspace (e.g., Figure 2).	
	Each subspace is a associated with a projection operator.	
	Projection operators (projectors) take the state vector and lay it down a subspace.	For example, we would have a projector for whether Linda is a bank teller.
	Squared projection length = probability.	
Basis sets	A basis set is a set of vectors which span a subspace.	In Figure 2, the BT, ~BT vectors are a basis set for the question of whether Linda is a bank teller.
	Each vector in the basis set corresponds to a question outcome.	
	Basis vectors can be used to create corresponding projectors.	In Figure 2, the BT basis vector can be used to define a projector for this subspace, which in turn can be used to compute the probability that Linda is a bank teller.
Unitary evolution and Schroedinger equation	The dynamical evolution of a quantum system, when it does not interact with its environment.	The dynamics typically reflect perpetual oscillations.
Open systems evolution	The dynamical evolution of a quantum system, when there is interaction with its environment.	Amplitudes/ probabilities stabilize. They have been rarely used in psychology.
Entangled state	When a composite system cannot be decomposed into the tensor product for the constituent systems.	Consider a composite concept, like pet fish. Can the meaning of the composite concept be produced by independently combining the states for pet and fish? Not if the composite system representation is
Supercorrelation, Bell inequalities.	The Bell inequality is a test of whether a quantity based on the expected values of two binary questions for each of two systems is below a certain limit or above. In the latter case we say that the systems	

Table 1. A summary of the main terms in quantum theory.

3. How can quantum structure emerge from a classical brain?

We could also ask 'how can classical probabilities emerge from the brain'. There is no more mystery for how quantum probabilities emerge at the cognitive level, than for e.g. classical probabilities. All the applications considered in this work employ quantum theory as a set of computational principles to build cognitive models. In the same way quantum processes can be programmed on a classical computer, so we assume that quantum processes at the cognitive level can emerge from classical brain neurophysiology. For example, superpositions at the cognitive level (i.e., the level of considering cognition/ behavior, without reference to neuronal processes) are considered epiphenomenal and their quantum interpretation is valid only at the cognitive level (Yearsley & Pothos, 2014).

Nonetheless, some researchers have attempted to specify the neurophysiological or otherwise origins of quantum structure in cognition. One idea relates to Suppes et al.'s (2012) neural oscillator proposal for how oscillation patterns involving synchronized neurons can correspond to stimulus, response associations. Note, there has been evidence that synchronization of firing rates amongst neurons can be related to cognitive processing (Eckhorn et al., 1988; Friedrich et al., 2004). Neural oscillators have wave-like properties and so can produce interference patterns, arising from phase differences between oscillators. De Barros (2012; de Barros & Suppes, 2009) argued that such interference patterns can produce the dynamics required for quantum models (they did this specifically for quantum models of question order effects, e.g., Wang & Busemeyer, 2013; see also Khrennikov, 2011). One question for such proposals is whether there is evidence for quantum structure beyond the observation that wave-like behavior can produce interference.

Busemeyer et al. (2015) showed that quantum computations can be implemented with a standard neural network. The neural network implemented three steps, a computation of amplitudes from unitary evolution, the conversion of amplitudes to probabilities, and state reduction. Notwithstanding the input and output mapping capabilities of neural networks (Churchland, 1990), this work provides an illustration of how a familiar, algorithmic framework can produce quantum-like output.

Beim Graben and Atmanspacher (2006; Atmanspacher & Scheingraber, 1987) considered how incompatibility between classical questions can sometimes emerge, if the description of the questions is coarse, so that questions have some irreducible fuzziness regarding possible outcomes. One issue is whether the kind of coarseness required to produce incompatibility arises in the way questions are mentally represented.

Finally, Aerts and Sassoli de Bianchi (2015) proposed that the quantum probability rule in cognition arises from averaging. Consider a typical decision experiment, in which participants are asked to assign probabilities to events. Suppose that different participants employ different probability rules – note, it is unlikely that each participant will have his/her idiosyncratic system for assigning probabilities to events, nevertheless more plausible scenarios can be envisaged as special cases of this general one. Invariably, the researcher analyzing the data averages results across participants. Are there expectations for what would be the form of such an average? According to Aerts and Sassoli de Bianchi (2014) such an average will be equivalent to the quantum rule for probabilistic assignment and this is why quantum theory appears successful in cognitive applications.

4. Are quantum cognitive models more complex than classical ones?

In quantum probability theory there are two types of questions (incompatible, compatible) and only one in classical theory (compatible). This fact may tempt the inference that quantum theory is all of classical theory (for compatible questions) and a little bit more (for incompatible ones), so that quantum cognitive models are necessarily more complex than classical ones. This is incorrect. Quantum cognitive models typically employ incompatible questions so that the complexity comparison involves a quantum model with incompatible questions and a broadly matched classical one. The issue of complexity concerns then how well constructed the quantum/ classical models are. On the few occasions when this has been studied in detail, the results favored the quantum model. Busemeyer et al. (2015) were the first to employ a Bayesian method to quantitatively compare matched, classical models for data on dynamic consistency (which concerns whether decision makers follow through with plans made in advance) and the comparison favored the quantum model. Trueblood et al. (2017) compared a hierarchy of models in a causal inference situation, including fully quantum and fully classical ones, with deviance information criterion (DIC), and analogously for Trueblood and Hemmer (2017) for results concerning episodic memory – in both cases quantum models were favored. A related issue concerns parameter recovery and quantum models have yet to be evaluated in this respect (for a promising demonstration see Mistry et al., in press). One possible issue is that for some models linear transformations on parameters can give the same output, because invariably probabilities are sinusoidal functions (so that adding 2π does not alter results).

Beyond comparisons between specific quantum, classical models, Atmanspacher and Romer (2012) attempted to provide a general complexity analysis (focused on parameter numbers) concerning quantum and classical models for question order effects, where the number of question outcomes and questions can vary. They observed that as this number increased, a generic quantum approach would be increasingly less complex than a matched classical one, because in the former case incompatibility can keep the overall representational dimensionality low, but not in the latter case.

Overall, there is no evidence that quantum cognitive models are systematically more complex than classical ones and some arguments (like Atmanspacher & Romer's, 2012) to the contrary.

5. Empirical research

The main objective of this work is to critically consider the range of cognitive applications of quantum theory. We have identified notable quantum models across perception, memory, similarity, conceptual processes, causal inference, constructive influences in judgment, decision order effects, conjunction/ disjunction fallacies in decision making, and other judgment phenomena. The finer categorization relating to decision making simply reflects the focus of quantum models and even so there is large variance in section size. Note, these divisions are for convenience of exposition. For each empirical domain, we will consider the relevant psychology, describe the quantum model with a focus on the aspects of quantum theory carrying the explanatory burden, and present a critical evaluation and any controversy. One issue will concern whether a quantum model is aimed at providing an ostensibly better explanation for previously known results or whether they have been employed in a generative way. This point deserves a few remarks. The main way in which quantum models were introduced in psychology concerns probabilistic results, such as the conjunction fallacy, which are incorrect with baseline classical probability inference, but could be shown correct with quantum inference (Busemeyer et al., 2011; Pothos & Busemeyer, 2009). It is worth appreciating the significance of this point: a result 'impossible' from a classical perspective, such as the conjunction fallacy, would be shown 'correct' when employing quantum probabilities. An important contribution of the quantum cognition programme is exactly that it revealed an alternative intuition for probabilistic inference and correctness (but note, correctness does not imply being normative, which is a separate issue, Pothos et al., 2017). Notwithstanding this point, results such as the conjunction fallacy are now nearly 35 years old, and there is an onus on the quantum cognition programme to reveal generative value too.

The final preliminary point is that certain empirical findings for which quantum models are implicated have attracted intense debate and it is difficult to make full justice to each of these debates. Inevitably our coverage will be selective, focused on the present aim, which is evaluating the contribution of quantum models.

5.1 Perception

5.1.1 Relevant psychology

We will consider two studies in this section, both relating to bistable perception. Bistable perception occurs with ambiguous figures, such as the Necker cube, which can be perceived in one of either two ways, such that each way reveals a different interpretation of the figure. Typically there is a sensation of effort in switching between interpretations.

Conte et al. (2009) employed a paradigm involving the sequential presentation of two ambiguous figures (each figure could be perceived in two different ways) or the presentation of just one of the figures. It is possible that seeing one figure first may result in some bias in perceiving the second figure and indeed Conte et al. (2009) reported a violation of the law of total probability, so that $p(A_+ \wedge B_-) + p(A_+ \wedge B_+) \neq p(A_+)$ (*A* and *B* refer to the two figures and the + and – signs to the two possible ways of perceiving them).

Atmanspacher and Filk (2010) considered the consistency of changes in bistable interpretation across different time points. Let us first define three quantities, $N^-(t_1, t_3)$, $N^-(t_2, t_3)$, and $N^-(t_1, t_2)$, which correspond to the number of cases in which the stimulus interpretation is different across the referenced time points. Set theory requires that $N^-(t_1, t_3) \leq N^-(t_1, t_2) + N^-(t_2, t_3)$, which is a form of the temporal version of the Bell inequality. Atmanspacher and Filk (2010) outlined the kind of empirical tests which would be required to demonstrate a violation of such a temporal Bell inequality, based on manipulations which ostensibly lead to interpretation switches with time.

5.1.2 **Quantum cognitive models**

Conte et al. (2009) argued that the violation of the law of total probability in their bistable perception results indicates that the mental states corresponding to the interpretation of each figure are superpositions, so that interference effects can arise. They presented a simple quantum model which provided good empirical coverage.

Atmanspacher and Filk's (2010) quantum model for bistable perception specifies the dynamics of processing a bistable stimulus focusing on how the decay of the mental state interacts with the dynamics of observing the stimulus. The model involves two components, one corresponding to changes in the mental state when the stimulus is not observed and another, called a cognitive update process, changing the state as a result of observation. The decay process is associated with the Hamiltonian and the observation one with the determination of stimulus interpretation. As long as the Hamiltonian and the observation operator do not commute (since if they do we have trivial dynamics), it is in principle possible to violate the temporal Bell inequality. Atmanspacher and Filk (2010) presented an entangled mental state for bistable perception and suitable Hamiltonian, observation operators, which can lead to violations of temporal Bell (the correlation between t_1, t_3 could exceed those across intermediate time points).

5.1.3 Critical evaluation and controversy

For Conte et al. (2009), there is a general point to make relating to any result ostensibly inconsistent with classical probability theory, to include violations of the law of total probability (conjunction, disjunction fallacies; the disjunction effect), question order effects, and any contextuality effects. All these results can be reconciled with classical probability theory through appropriate conditionalization. For example, regarding the conjunction fallacy, one could write $Prob(A\&B|fm1) > Prob(A|fm2)$, where *fm1*, *fm2* can correspond to different frame of minds (Dzhafarov & Kon, in press). However, one hardly ever encounters such arguments, because an arbitrary, post hoc conditionalization carries low explanatory power. We will therefore automatically discount such possibilities in subsequent discussion.

Note that classically conditionalizing on e.g. frame of mind indicates an assumption that mental processing is contextual and contextuality is essentially how quantum theory provides an account of such results too: if the A, B questions are contextual, processing the A question first creates a context for the subsequent B question which is different from that when the B question is considered first. Kujala and Dzhafarov (2014, p.2) noted for contextual variables that ''these random variables cannot be sewn together into a single system of jointly distributed random variables if one assumes that all or some of them preserve their identity across different conditions". Quantum theory is hardly unique in incorporating contextual influences. An advantage of the quantum approach is that there are specific rules for relating different probability spaces arising from contextually (quantum theory can be thought of as a classical probability framework, but where there is a need to integrate together multiple probability spaces; e.g., Hughes, 1989).

Atmanspacher and Filk's (2010) work makes a bold prediction: in order to violate the temporal Bell inequality, it has to be the case that participants do not switch interpretation across t_1, t_2 and also do not switch between t_2, t_3, but then switch across t_1, t_3, which violates an obvious intuition of transitivity. The only plausible explanation for such a pattern of results would be that the relevant mental state is extended time-wise, that is, it does not have a well-defined trajectory through time. A timewise non-local mental state may extend, for example, across both periods t_1, t_2, so that one can no longer assume that an event at t_1 caused changes at t_2. So, if the predictions from this work were to be empirically confirmed, this would have implications for our understanding of causality across time (Yearsley & Pothos, 2014).

Precise predictions from Atmanspacher and Filk's (2010) framework for when the temporal Bell inequality is likely to be violated depend on the details of how the Hamiltonian and observation operators are specified. Further work is needed to motivate (ideally pre-fit with pilot data) the various components of the quantum model. Another issue is how tightly can we make an association between a putative violation of temporal Bell and quantum processes. We fully discuss this issue in Section 5.4.2 but, briefly, even a classical system with disturbing, 'clumsy' measurements can lead to violations of Bell. Therefore, before a violation of temporal Bell can be uniquely associated with quantum structure at the cognitive level, a number of preconditions need to be tested (Wilde & Mizel, 2012).

5.2 Memory

5.2.1 Relevant psychology

We first consider Bruza et al. (2009) who outlined an empirical set-up suitable for testing violations of the Bell inequality in cued recall memory. Consider a cued recall experiment involving cues that can have multiple senses, e.g., the cue "bat" can have an animal and a sports sense. Study is divided into parts, so that in each part a cue word is presented with words which activate different senses (e.g., in one study part the cue bat might appear with the words ball and glove, so as to activate the sports context). Post study, two cue words are presented individually with a request to recall other words from the lists just studied, with a view to examine the word senses activated by the cues, by interpreting the recalled words. The authors outline a 2x2 design of cues (two cues, each of which has two senses), which can potentially lead to violations of the Bell bound.

A number of researchers have explored a so-called memory overdistribution effect and variants in memory recognition. In the typical paradigm, participants encode a set of memory targets, for example, a word list. In test participants are presented with the targets, related distractors that are semantically related to the targets, and unrelated distractors. Recognition instructions can be varied factorially with the test items, for example, accept related distractors but reject targets and unrelated distractors. If T symbolizes targets and R related distractors and probes are included for the disjunction and the marginals, then a key empirical finding is that recognition probabilities indicate $Prob(T) + Prob(R) > Prob(T \cup R)$, which is a disjunction fallacy (also called episodic overdistribution). Classically, since the categories of T and R are mutually exclusive, this is impossible, and we instead require $Prob(T) + Prob(R) = Prob(T \cup R)$. This empirical finding illustrates that some items are being remembered as both presented and not presented (Brainerd & Reyna, 2008; Brainerd et al., 2010). Together with the disjunction fallacy, we also have a subadditivity effect, which is when the probability of accepting an item in mutually exclusive and exhaustive categories is greater than one (so, subadditivity also indicates $Prob(T) + Prob(R) > Prob(T \cup R)$, but in this case there is no overt $T \cup R$ cue).

5.2.2 Quantum cognitive models

Bruza et al. (2009) aimed to illustrate that a quantum representation based on an entangled state, matched to the assumptions of their paradigm, allows for violations of the Bell bound. All words were represented as rays.

Regarding the memory overdistribution effect, Brainerd et al. (2013) developed a quantum representation model, for subadditivity in a particular three list variant of the recognition paradigm (Brainerd et al., 2012). Each item in the empirical test was represented in a five dimensional Hilbert space, such that three dimensions corresponded to verbatim features (mostly superficial characteristics) for each of the three lists, one dimension to gist features (mostly semantic/ abstract characteristics, but also relational, contextual information), and one dimension to distractor features. The model in the Brainerd et al. (2013) work was termed the QEM model (quantum episodic memory model). Brainerd et al. (2015) provided an elaboration of the Brainerd et al. (2013) model and presented it as a formalization of fuzzy trace theory (Reyna, 2008; Reyna & Brainerd, 1995), which we call QEM+, abusing notation for simplicity. In the QEM+, each item is represented in a three dimensional Hilbert space, with basis vectors corresponding to verbatim, gist, and other information (information not matching the cue's either surface or semantic content). According to QEM+, subadditivity is directly predicted because the retrieval probabilities are computed in a way that the contribution from the gist information appears twice in verbatim questions. Note that the QEM+ could apply either for incompatible or compatible memory measures (e.g., regarding the various probes), creating a need for determining compatibility (Section 7).

Denolf and Lambert-Mogiliansky (2016) noted that in the QEM/ QEM+ models verbatim information, gist information etc. are represented with orthogonal vectors, which makes them mutually exclusive (i.e., completely accepting gist information for a test cue means completely rejecting verbatim information). Instead, Denolf and Lambert-Mogiliansky (2016) proposed a quantum memory model called Complementary Memory Types (CMT) in which the verbatim information and gist information are modeled as incompatible, so that they not commeasurable and perfect knowledge of one introduces some uncertainty for the other. Denolf and Lambert-Mogiliansky model (2016) made a symmetry assumption for the representation of the gist information. In a four dimensional space, three basis vectors corresponded to verbatim information for each one of the three memorized lists (in the experiment covered by the model) and one to unrelated information. An alternative basis set corresponded to gist information. Gist information was modeled with a single vector, with equal amplitudes along the basis vectors for each memorized list and zero for unrelated information.

An approach based on incompatibility was further developed by Trueblood and Hemmer (2017; see also Broekaert & Busemeyer, 2017). In their Generalized Quantum Episodic Memory Model (GQEM), verbatim, gist, and new information are assumed incompatible. Each of the three types of information corresponds to a two-dimensional basis, obviating the problem of differing dimensionalities in Denolf and Lambert-Mogiliansky (2016). Acceptance probabilities under verbatim and gist instructions are computed by first evaluating acceptance based on gist information and, if this produces a negative outcome, evaluating acceptance on verbatim information (gist information has been argued to be processed more rapidly; Brainerd et al., 1999). An analogous assumption was employed with paradigms in which training items are presented in different contexts.

5.2.3 **Critical evaluation and controversy**

Bruza et al.'s (2009) work awaits empirical examination, which if provided would indicate that it is not possible to have a four-way classical probability distribution for the different senses of the recall words.

The disjunction fallacy/ subadditivity in memory recognition have challenged standard recognition models, such as the one-process signal detection model (Glanzer & Adams, 1990), and standard source recognition models, such as the source monitoring model (Batchelder & Riefer 1990), which unelaborated cannot predict these effects. By contrast, Brainerd et al. (2013) reported acceptable fits for the QEM for the Brainerd et al. (2012) data, when compared to their own predominant model for the episodic subadditivity effect (the Overdistribution model). Brainerd et al. (2015) reported results supporting predictions from their QEM+, notably that it is specifically subadditivity and not superadditivity that is predicted and also that empirical manipulations which enhance reliance on gist information and/ or decrease influence from verbatim information should both increase subadditivity. Both QEM and QEM+ are elegant in their parsimony, but make limited use of quantum features questioning the necessity for quantum theory. For example, Brainerd et al. (2015) noted that a key assumption in fuzzy trace theory is that gist information allows the same item to be perceived as belonging to different categories, which is why in the QEM+ gist information appears twice in retrieval probabilities, producing subadditivity.

Denolf and Lambert-Mogiliansky's (2016) and Trueblood and Hemmer's (2017) proposals that fuzzy trace theory can be elaborated with an assumption of incompatibility between verbatim, gist information is arguably a more significant contribution of quantum models in this area. However, Denolf and Lambert-Mogiliansky's (2016) symmetry assumption is hard to justify.

Denolf and Lambert-Mogiliansky (2016) reported coverage of both subadditivity and favorable comparisons with QEM+. Trueblood and Hemmer (2017) noted some problematic predictions from the QEM+. For example, retrieval probabilities are predicted to be equal under verbatim and verbatim plus gist instructions, but empirical results show the latter to be higher. Moreover, the QEM+ predicts a higher probability of accepting a test cue with verbatim instructions, compared to gist instructions, again contrary to empirical data. Regarding the Overdistribution model, they noted that the model can cover the episodic overdistribution effect, but not the subadditivity effect (the model is underspecified regarding acceptance probabilities under unrelated new instructions). They conducted a hierarchical Bayesian model comparison between their GQEM model and the Overdistribution model, based on a new experiment using the item memory variant of the memory task (test cues can be old or new) and results from Kellen et al. (2014) using the source memory variant (training cues presented in different contexts). The former comparison favored the GQEM. The latter comparison favored the Overdistribution model, but GQEM was shown to produce equivalent fits to the data.

5.3 Similarity

5.3.1 Relevant psychology

Geometric models of similarity whereby objects are represented as points and similarity is some function of the distance between them have been hugely influential (e.g., Nosofsky, 1984; Shepard, 1987). Tversky (1977) reported some key challenges to such models. He argued that similarity judgments can violate minimality, symmetry, and the triangle inequality as well as being subject to contextual influences from the range of stimuli concurrently considered (a diagnosticity effect). He proposed that violations of symmetry arise from differences in prominence/ degree of knowledge, so that e.g. $Sim(\,China, Korea) < Sim(\,Korea, China)$, where participants are assumed to have more knowledge of China than Korea. For violations of the triangle inequality he used William James's example, whereby *Similarity*(Russia, Jamaica) is low, but *Similarity*(Russia, Cuba) is high (because of political affiliation) and *Similarity*(Cuba, Jamaica) is also high (because of geographical proximity), so that *Similarity*(Russia, Jamaica) < *Similarity*(Russia, Cuba) + *Similarity*(Cuba, Jamaica). Finally, an example of the diagnosticity effect is that out of Hungary, Sweden, Poland, the country chosen as most similar to Austria is Sweden, but out of Hungary, Sweden, Norway, it was Hungary.

Violations of symmetry have been demonstrated on multiple occasions (e.g., Aguilar & Medin, 1999). However, despite the strong intuition for the triangle inequality, Tversky's (1977) example concerns violations of the triangle inequality on dissimilarities, not similarities. That is, the triangle inequality constraint (from an assumed metric representation) is that *Dissimilarity*(Russia, Jamaica) < *Dissimilarity*(Russia, Cuba) + *Dissimilarity*(Cuba, Jamaica), where dissimilarity can be straightforwardly associated with distance. It is not straightforward to translate an inequality on dissimilarities to one on similarities. Tversky seemed to be aware of this issue (cf. Tversky & Gati, 1982), but it has been ignored in much of subsequent literature. Considering options for how to convert the triangle inequality on dissimilarities to similarities, Yearsley et al. (2017) derived a so-called multiplicative triangle inequality (based on Shepard's, 1987, function for relating distances to similarities). Additionally, replications of the diagnosticity effect have been rare (Evers & Lakens, 2014).

Finally, even though much of the similarity literature has developed in terms of point-wise comparisons, researchers have recognized that matching parts between object representations can have a greater influence on similarity judgments, than mismatching parts (e.g., Gentner, 1983; Goldstone, 1994).

5.3.2 Quantum cognitive models

A prior motivation for applying quantum theory in similarity is that it involves geometric representations and geometric representations have consistently featured prominently in similarity models (Nosofsky, 1984; Shepard, 1987). But, whereas in traditional geometric models representation is point-wise (each object corresponds to a single point), in quantum models representations can be subspaces. So, quantum models have a natural way to capture differences in knowledge between concepts. Additionally, quantum probabilities embody order effects and are contextual, which are baseline requirements respectively for violations of symmetry and the diagnosticity effect.

Pothos et al. (2013) modeled similarity judgments as quantum conjunctive probabilities, involving a projection order matching the order in which the compared stimuli are referenced and an initial state which is neutral, that is set so that when comparing stimuli A, B, we have $|P_A \cdot |\psi\rangle|^2 = |P_B \cdot |\psi\rangle|^2$. Then the model immediately produces e.g. $Sim(\ China, Korea) < Sim(\ Korea, China)$, as long as the dimensionality of the subspace representing China is greater than the one for Korea. This was proved in Pothos et al. (2013) for two-dimensional vs. one-dimensional subspaces and arguments were offered as to why this result generalizes. Regarding violations of the triangle inequality, different regions in a quantum space are associated with different contexts (e.g., features/ concepts). So, using Tversky's (1977) example, if a region of a quantum space is consistent with the property of Communism, then representations in or close to that region would reflect varying degrees of consistency with Communism. Therefore, if basis sets are arranged in a two-dimensional space so that one region reflects Cuba, sandwiched between a region for Russia and one for Jamaica, it is straightforward to see how triangle inequality violations can occur (at least on dissimilarities). Coverage of the diagnosticity effect relies on order effects in non-commuting projectors. Pothos et al. (2013) assumed that each possible pairwise similarity was computed with the quantum similarity rule, but modified to include prior projections to any context stimuli. So, in a sequence of projections involving both the compared stimuli and the context ones, differences in eventual selection were observed consistent with the diagnosticity effect.

Regarding similarity comparisons sensitive to structure, Pothos and Trueblood (2015) adapted the quantum similarity model using the idea of Smolensky (1990) for structure in linguistic representations based on tensor products. For example, for stimuli having three parts (top, middle, bottom), such that each part can differ in color and shape, we can write a representation vector as $|top\ part\rangle \otimes |shape\rangle \otimes |color\rangle + |middle\ part\rangle \otimes |shape\rangle \otimes |color\rangle + |bottom\ part\rangle \otimes |shape\rangle \otimes |color\rangle$. The 'part' vectors keep track of matching parts (and can be adjusted to capture influence from matches in place or matches out of place, Goldstone, 1994). For example if $|top\ part\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|middle\ part\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ etc. then in computing similarity there are contributions only across matching parts; briefly this is because $\langle x \otimes y \otimes z | x' \otimes y' \otimes z' \rangle = \langle x|x'\rangle \langle y|y'\rangle \langle z|z'\rangle$, that is tensor product structure respects orthogonality in each space individually.

5.3.3 **Critical evaluation and controversy**

Pothos et al.'s (2013) assumption of a mental state that is neutral in relation to the compared stimuli was justified as one of uniformed priors. For the diagnosticity effect, incorporating contextual influence as prior projections takes advantage of quantum theory's contextual features, in that the same question has to be considered differently depending on whether it is considered in isolation or in the context of prior incompatible questions; note the final projection depends on the entire projection sequence. However, there is an issue of scalability when dealing with multiple contextual items.

One can question a representation scheme which assumes that concepts are represented as incompatible. The rationale is that a sequence of projections corresponds to a train of thought, so that successive projections correspond to the mental state going through different concepts. Depending on which concept is currently considered, incompatibility means that there are differing perspectives on the likelihood of thinking about other concepts. For example, if one is thinking about China there is uncertainty for whether he/she will next think about Korea, depending on the relation between the subspaces.

The main contribution of the quantum similarity model was to show how asymmetries in the extent of knowledge between concepts can be captured with differences in subspace dimensionality and that this produces similarity asymmetries in the observed direction. Even though the model has not had generative value, the completeness of coverage of Tversky's (1977) seminal results compares favorably to that from other predominant approaches. For example, the main issue with Tversky's (1977) contrast model is that it involves two independent parameters. For violations of symmetry, these had to be set in a circumvented way (based on assumptions about the relative weighting of the features of the subject and the referent in a similarity comparison). Krumhansl's (1978, 1988) distance-density model encounters a problem regarding violations of symmetry, since adequate coverage requires the assumption that China is similar to more other countries than Korea. Additionally, regarding the triangle inequality, the model incorporates a mechanism of computing similarity within reduced subspaces, corresponding to different contexts, but this is underspecified (e.g., how to determine such subspaces). For symmetry, Ashby and Perrin's (1988) General Recognition Theory required an assumption of how many other counties are similar to Korea vs. China *opposite* to that of Krumhansl (1978)[2]. Additionally, for violations of the triangle inequality, inequivalent perceptual distributions were required for Russia, Jamaica, and Cuba, and there are no a priori reasons to anticipate such an assumption.

Regarding Pothos and Trueblood's (2015) proposal for structural similarity, currently this is descriptive and its theoretical merit rests in revealing a common framework for structure between language (as in Smolensky, 1990) and similarity.

5.4 Conceptual reasoning

5.4.1 Relevant psychology

It is sometimes the case that a particular instance can be a good example of a composite concept, but a poor example of either individual concept. For example, a goldfish is a good example of the combined concept pet-fish, but a poor example of either pet or fish (e.g., Hampton 1988a, b; Osherson & Smith, 1981; Storms et al., 1999). Specifically, an overextension effect is when the strength of category membership for a combined concept is greater than for other individual concepts; and an underextension effect occurs when the converse is true. Both these effects indicate violations of the law of total probability. Analogous effects have been observed for disjunctive conceptual combinations.

Bruza et al. (2015) addressed the question of whether there is compositional structure in conceptual combination. They employed novel, ambiguous conceptual combinations composed of two words, such that each word had two (fairly) distinct meanings, e.g., in 'boxer bat', boxer can refer to a person or a dog and bat to a sporting equipment or an animal. Each participant received one of four primes (a single word), corresponding to the 2 x 2 meaning possibilities, and subsequently was asked to interpret the ambiguous concept and rate the sense of each component word that was employed. The authors then compiled conditional probabilities for the interpretation of each concept combination, given particular primes. These probabilities can be employed to test for quantum entanglement, with a variant of the CHSH inequality. Violations of CHSH were observed, indicating non-compositionality, that is, the semantics for the conceptual combination cannot be reconstructed by looking at the semantics of each component concept independently.

Non-compositionality in concept associations (rather than conceptual combination as such) was further explored by Cervantes et al. (in press). Participants were presented with one A question and one B question, such that A1: Gerda or Troll, A2: Snow Queen or Old Finn woman, B1: Beautiful or Unattractive, B2: Kind or Evil. Participant responses were used to compute expectations e.g. $E[A1, B1]$,

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[2] Ashby and Perrin (1988, p.133) noted that "…for many people North Korea is very similar to several other countries." But, recall, Krumhansl (1978, p.454) made the exact opposite assumption, "If prominent countries … are those stimuli having relatively many features, then these objects have features in common with a larger number of different objects….". In other words, Krumhansl (1978) assumed that it is China, not Korea, which is similar to a greater number of other countries.

by coding question answers with ± 1. Cervantes et al. (in press) reported violations of the CHSH, hence demonstrating non-compositionality in concept associations of this kind. Note, the conclusion of supercorrelation follows by considering the relation between different pairs of *A*, *B* questions.

A challenging issue in conceptual reasoning concerns borderline vagueness, the idea that for many concepts there are no clearly defined boundaries and there are borderline cases for which it is unclear whether the predicate applies. The Sorites paradox exemplifies such problems: Start by stating that if X is a heap of sand, then removing one grain will surely result in a heap. However, the repeated application of this rule results in the paradox that the last grain left must still count as a heap. The present focus is on the acceptance of borderline contradictions along the lines such as 'X is tall and not tall', where X refers to a borderline case (Alxatib and Pelletier, 2011).

5.4.2 Quantum cognitive models

We focus on Aerts's (2009; see also Aerts & Gabora, 2005a, 2005b; Aerts, Sozzo, & Veloz, 2015) model for conceptual application, which is a rare cognitive application of quantum field theory and Fock spaces. A Fock space is a superposition of different Hilbert spaces. For a conceptual combination between concepts A, B, Aerts (2009) proposed a part based on the tensor product between the (quantum) representations for *A*, *B* and a part based on superposition, so that the resulting state would be given by $\psi(A,B) = me^{i\theta}|A\rangle\otimes|B\rangle + \frac{ne^{i\phi}}{\sqrt{2}}(|A\rangle + |B\rangle)$, with $m^2 + n^2 = 1$. The Fock space structure is essentially $X\otimes Y \oplus (X + Y)$, where *X*, *Y* are states. Note, this combination rule is intended for conjunctions and the one for disjunctions follows from this one. The model purports to encompass a part corresponding to classical conceptual combination (the tensor product space) and a non-classical part (the superposition). Interference effects arise from the superposition part, which allow the modelling of overextensions and underextensions. This model purports to formalize the idea of two separate routes to concept combination, a classical one and a quantum one.

Regarding non-compositionality in conceptual combinations, Bruza et al.'s (2015) application of quantum theory more broadly relates to the formalization of corresponding tests from the quantum literature (Bell, 2004). Bruza et al. (2015) provided examples of non-compositional quantum representations for conceptual combinations.

Cervantes et al. (in press) generalized the CHSH inequality in a way that possible influences from signaling can be discounted. The idea is that exceeding the CHSH bound can arise from signaling, which is the extent to which the measurement of e.g. A1 is disturbed by the measurement of B1 vs. B2, quantified as the difference in the marginal distribution of A1 in each of the two measurements situations (B1 vs. B2). When these marginal distributions are the same, then we can assume that the measurement context is not disturbing, which Cervantes et al. (in press) call the no-signaling or marginal selectivity situation. Under such circumstances, according to Cervantes et al. (in press) we have a purer test of contextuality, to mean altering the meaning of the relevant concepts (e.g., kind vs. evil), depending on context (e.g., which characters are being evaluated; Dzhafarov et al., 2015, 2016; Kujala & Dzhafarov, 2013, 2014; cf. Wilde & Mizel, 2012). The expression for the generalized CHSH inequality is

$$max_{k,l\in\{1,2\}}\left|\sum_{i,j\in\{1,2\}} E\left[A_i^j B_j^i\right] - 2E\left[A_k^l B_l^k\right]\right| - \sum_{i\in\{1,2\}}\left|E\left[A_i^1\right] - E\left[A_i^2\right]\right| - \sum_{j\in\{1,2\}}\left|E\left[B_i^1\right] - E\left[B_i^2\right]\right| \leq 2$$

where *E* indicates expectation.

Blutner et al. (2013) developed a quantum model where probabilities corresponded to activation patterns in a Hopfield network with a single input unit representing the statement of interest (e.g., "John is tall") and two output units representing truth, falsity. The statement could be characterized in one of three ways, definite truth, definite falsity, and truth and falsity at the same time – this last pattern expresses a 'glut' and is a feature of borderline vagueness. Probabilities were assigned to each of these three possibilities using the Boltzmann distribution. The main assumption was that the projectors for truth and falsity were not orthogonal, allowing emergence of interference terms, which affected the normalization factors for the Boltzmann distribution probabilities.

5.4.3 Critical evaluation and controversy

Aerts (2009) reported close fits to overextension, underextension effects in conceptual combination, covering both conjunctive and disjunctive categories (mostly using Hampton's, 1988a, 1988b, data), though the number of model parameters does not compare favorably to data degrees of freedom. Standard conceptual combination approaches have difficulty with such effects. For example, Hampton (2011) discussed fuzzy logic averaging approaches to overextensions, underextensions, and negations to conclude that they fail to appropriately capture interactions between concepts and, instead, an approach based on prototypes is more promising. His prototypes approach involves various heuristic assumptions and so a more formalized approach (such as Aerts's, 2009) may provide a tighter explanation.

A critical point is that the forms employed for conjunction and disjunction and the postulated relation between category membership in the conjunctive concept and in the disjunctive concept (the probability of membership in a disjunctive concept is one minus that for a conjunctive concept) do not cleanly match expectations for such operations. This issue also arises in Aerts's (2009) application of his conceptual combination model to the disjunction effect, which is a violation of the sure thing principle in decision making (Tversky & Shafir, 1992; Section 5.8). The two concepts were equated with the various conditions required for testing the sure things principle. This creates a confusing picture, since in a conceptual combination we have $Membership(A\&B) = f\big(Membership(A), Membership(B)\big)$. However, for the disjunction effect we require $Probability(action) = Probability(action\&known\ X) + Probability(action\&known\ Y)$. Despite these qualifications, Aerts's (2009) model formalizes a dual route for conceptual combination and it has further value as one of the pioneering quantum cognition models (in Aerts & Gabora, 2005a, 2005b).

Bruza et al. (2015) identified some conceptual combinations for which there was evidence of non-compositionality, regardless of the origins of this non-compositionality, i.e., contextuality vs. disturbing measurements. These results impact on approaches to conceptual combination requiring compositionality (e.g., Fodor's, 1994), less so on ones for which a compositional constraint is less central, e.g., proposals based on emergent properties (Hampton, 1997). Non-compositionality and weaker forms of compositionality are now well established in the conceptual combination literature (Swinney et al., 2007). So, the contribution of Bruza et al. (2015) is providing a formal framework for assessing compositionality/ contextuality. A demonstration of contextuality, however, does not necessitate a quantum model. Also, one can envisage a model with an entangled state (perhaps analogous to that in Bruza et al., 2015) and consider the intriguing question of what would be the generative value of such a model.

Cervantes et al. (in press; Dzhafarov et al., 2015) argued that there is no evidence for pure contextuality in any psychological demonstration of the CHSH inequality, apart from Cervantes et al. (in press). They were the first to demonstrate that contextuality is possible in behavior, without disturbing measurements/ judgments, with the help of their technically sophisticated elaboration of the CHSH inequality.

Blutner at al. (2013) showed that Alxatib and Pelletier's (2011) experimental data could be fit by their quantum model, but not a closely matched classical one (for which probability assignment was identical to the quantum one, with the exception of the normalization factor for which interference terms arose in the quantum case). So, this is another example of how interference terms in quantum probabilities are employed to accommodate effectively a violation of the law of total probability.

5.5 Causal inference

5.5.1 Empirical research

This section concerns the way knowledge of the causal structure linking causes and effects for a particular situation, and some related information, affects inference. Trueblood at al. (2017) presented three experiments, broadly based on the causal structures and procedures from Rehder (2014), seeking to challenge classical intuition in causal inference. They reported three novel empirical findings. First, they observed evidence for reciprocity, that is, $Prob(A|B) = Prob(B|A)$, which is surprising because of the asymmetry between causes and effects (reciprocity shows insensitivity in causal direction). The inverse fallacy (Kahneman & Tversky, 1972) is related to reciprocity, but there have not been demonstrations in causal inference. Second, there was evidence for memorylessness, which occurs when conditionalization in probability estimation depends only on the latest information to be considered. For example, in some cases conditionalization on *A* & then *B* would appear equivalent to just *B*. Finally, they observed evidence that classical inconsistencies were more likely to be adopted in unfamiliar situations and for participants who approached the task in a more intuitive way (as opposed to deliberative, measured using the Cognitive Reflection Test; Frederick, 2005). Together with these novel empirical findings, Trueblood et al. (2017) also reported violations of the Markov condition, anti-discounting behavior, and order effects, partly replicating Rehder (2014).

5.5.2 Quantum cognitive models

The evidence for inconsistencies with classical probability principles in causal inference motivates the application of quantum theory. Equally, there is evidence for some classical behavior in causal inference (Rehder, 2014). Accordingly, Trueblood et al. (2017) specified a hierarchy of representations, from fully quantum to fully classical, which included intermediate levels with some effects/ causes represented in compatible ways and some in incompatible ways. The fully classical representation required an eight dimensional space, since all causal reasoning scenarios involved three binary variables, while the fully quantum representation required only a two dimensional space (cf. Trueblood & Busemeyer, 2012). The advantage of the hierarchical approach was that quantum and classical putative influences could be integrated in the same framework in a coherent manner. Regarding the quantum part of the model, different causes, effects were represented as rays and rotation matrices were fitted to the data to achieve empirical fits. Probabilities were computed by projectors in nearly all cases, except for the fully quantum one, for which a version with positive-valued operator measures (POVMs) outperformed a version with projectors. A POVM projects to a subspace, but whereas using a projector is noiseless, POVM projection is associated with a small error that projection will go to another subspace. That is, POVMs incorporate the idea that there may be other possibilities either causing, or be caused by, a particular event. In Trueblood et al.'s (2017) modelling, POVMs allowed the circumvention of reciprocity (which was not observed consistently).

5.5.3 Critical evaluation and controversy

Trueblood et al. (2017) employed model comparison techniques penalizing for number of parameters and so determined the most appropriate level of quantumness for describing the data, for each experiment. In all cases, a degree of quantumness was required for best fit. Prior to Trueblood et a.'s (2017) work, it had already been recognized that causal inference can violate classical principles, which undermined the then predominant descriptive and normative framework for causal reasoning based Bayesian networks (Pearl, 1988). Rehder's (2014) results provided strong evidence that, at least in some cases, there is a discrepancy between predictions from Bayes nets and human behavior (see also Fernbach & Sloman, 2009; Park & Sloman, 2013; Rottman & Hastie, 2016). Rehder's (2014) approach for the range of behaviors in causal inference, including both classical and non-classical biases, was to specify three heuristic models, which together with a Bayesian one could additively combine to predict participant behavior. Trueblood et al. (2017) argued that their main contribution was to show how individual heuristic principles were not needed, but rather non-classical behavior in causal inference could be explained by employing incompatibility/ interference effects from quantum theory (see also Mistry et al. in press). Note, even if there is no technical need for separate heuristics, heuristics potentially provide descriptive interpretations for subsets of the relevant behavior under quantum theory (Mistry et al., in press). Also, Trueblood et al. (2017) showed that more compatible representations were more likely to be observed for participants adopting a more analytic mode of reasoning (Frederick, 2005) and after more experience with the task, so supporting their view for the source of incompatibility in cognition.

5.6 Constructive influences in judgment

5.6.1 Empirical research

Sometimes an opinion or judgment appears to alter the underlying mental state (Brehm, 1956; Lichtenstein & Slovic, 2006; Schwarz, 2007; Sharot et al., 2010). For example, an earlier judgment can activate thoughts or perspectives that alter perception of subsequent ones. Or it is possible that a choice biases a re-interpretation of preferences to avoid cognitive dissonance (Festinger, 1957). For example, Glöckner et al. (2010) demonstrated coherence shifts, changes in subjective cue validities related to a decision, in a direction indicating greater consistency with the decision. Likewise, in a hypothetical legal case, Holyoak and Simon (1999; Simon et al., 2001) showed that the evaluation of arguments changed to become more consistent with the produced verdict. There have been some studies attempting to harness such intuitions into paradigms suitable for applications of quantum theory.

White et al. (2014, 2015) provided the most direct demonstration of constructive influences in judgment, as predicted by a quantum framework. They employed a paradigm of presenting two stimuli of opposite positive, negative valence (e.g., corresponding to imagined smartphone ads). The pairs of stimuli were always presented in the same order. In all cases participants provided an affective rating for the second stimulus. In some cases, participants provided an affective rating for the first one as well. White et al. (2014, 2015) observed what they called an evaluation bias, according to which the second stimulus was rated in a more extreme way when the first stimulus had been rated too, than when it had not. Put differently, the intermediate judgment created an impression of greater contrast between the first and second stimuli. White et al. (2014, 2015) presented several modifications to their procedure and controls, and generalized the finding with other kinds of judgments (e.g., trustworthiness).

Kvam et al. (2014) examined constructive influences in a prisoner's dilemma task (Shafir & Tversky, 1992). In such a task, participants have to decide whether to defect or cooperate with a (usually hypothetical) associate, and then a payoff is assigned to different combinations of actions (Section 5.8.1). The payoffs usually encourage defection, on an assumption of associate cooperation. Kvam et al. (2014) employed a sequential prisoner's dilemma task with the added manipulation that in-between the two tasks participants were asked to state whether they were intending to cooperate. They found that a statement of cooperation impacted behavior on the second prisoner's dilemma task.

The idea that decisions can force the identification of the mental state with the decision outcome was explored by Yearsley and Pothos (2016). They employed a hypothetical murder mystery, such that the suspect would be originally considered innocent. All participants would then receive 12 pieces of evidence indicating the suspect to be guilty, such that each piece of evidence was individually weak but collectively they would make a strong case of guilt. With a between participants manipulation, Yearsley and Pothos (2016) varied the number of intermediate judgments of guilt for the suspect. They found that increasing the number of intermediate judgments slowed down opinion change. Note, this is the quantum Zeno prediction in physical systems, as translated to opinion change.

In Kvam et al. (2015) participants were asked to judge the direction of motion in a dynamic dot display (for the subset of dots which were moving coherently) and provide confidence ratings too. The main empirical finding was that confidence ratings were less extreme when participants made an intermediate choice prior to rating confidence, than when they did not, as long as there was some time delay between the judgment and confidence ratings.

5.6.2 Quantum cognitive models

White et al.'s (2014, 2015) quantum model for the evaluation bias assumed a basis for positive, negative affect and an initial state close to one of these rays, depending on the valence of the first stimulus (e.g., if the first stimulus was positive, the mental state was close to the positive affect ray). The subsequent oppositely valenced stimulus was modeled with an appropriate rotation (e.g., if the subsequent stimulus was negative, the rotation was towards the negative ray). The impact of the intermediate rating was a projection of the mental state to the corresponding subspace, which meant that the subsequent rotation would bring the mental state closer to the oppositely valenced ray. Therefore, the collapse assumption in quantum theory drives prediction.

Analogously, Kvam et al. (2014) employed a quantum model for behavior in the prisoner's dilemma task, for how internal states of intention to cooperate or not relate to action, and so associated an intermediate statement of cooperativeness to performance in the second PD task. The intermediate statement of cooperation between the two tasks collapses the mental state.

Yearsley and Pothos (2016) employed a two-dimensional space, with a basis set corresponding to the guilt, innocence of the suspect. The effect of each piece of evidence on the mental state was modeled by a time-dependent unitary operator, so that the impact of particular pieces of evidence on opinion change depended on their serial position (Hogarth & Einhorn, 1992). Intermediate judgments on the mental state were modeled through POVMs, since with several judgments (in some conditions) there might be large potential error in any single judgment. Yearsley and Pothos (2016) computed an analytic expression for the survival probability (probability of no-change) as a function of intermediate judgments. The main feature of their quantum model is the collapse postulate.

Kvam et al. (2015) employed a quantum random walk model, involving a tridiagonal Hamiltonian, from the Feynman crystal model (Feynman & Hibbs, 1965). Off diagonal entries diffuse amplitude to adjacent states, which correspond to all possible confidence levels. The parameterization of the Hamiltonian was consistent with that of the intensity matrix for a matched classical Markov random walk model (Pike, 1966). Without an intermediate choice, amplitude was pushed towards extreme confidence levels. With an intermediate choice and subsequent processing of the stimuli (second stage processing), the flow of amplitude towards extreme confidence levels was less pronounced, as empirically observed. Interference arises in the model because the second stage processing makes the initially commuting projectors for judgment and confidence ratings noncommuting.

5.6.3 Critical evaluation and controversy

White et al. (2014, 2015) aimed to show how the projection associated with the intermediate judgment could account for a change in second stimulus evaluation, as predicted by quantum theory. No specific fits were carried out, but the finding of the evaluation bias was considered to confirm the quantum prediction, based on certain assumptions (e.g., the relative placement of rays, which White et al., 2014, justified with consistency arguments). White et al. (2014) reported that the evaluation bias was an a priori prediction of quantum theory. They examined Hogarth and Einhorn's (1992) anchoring and adjustment model, as an eventual judgment from multiple pieces of evidence can be influenced by intermediate judgments. However, White et al. (2014) showed that reasonable parameterizations of Hogarth and Einhorn's (1992) model do not allow the evaluation bias. Memory or attention processes could also potentially account for the evaluation bias. For example, the intermediate judgment potentially creates a stronger memory trace for the first stimulus, which then creates a stronger contrast with the second stimulus. White et al. (2014, 2015) argued that such potential explanations for the evaluation bias are complementary to the formal description from quantum theory.

Kvam et al.'s (2014) quantum model fitted their prisoner's dilemma data better than a matched classical model, which lacked a constructive influence from the intermediate judgment. Note, whether a quantum model can predict a contrast effect (as in White et al., 2014, 2015) or not (cf. Kvam et al., 2014) depends on the specific model assumptions (subspaces, initial state etc.); quantum theory can accommodate both possibilities.

Yearsley and Pothos (2016) compared the quantum model with a matched classical probability model, which also incorporated error-prone decisions; the two models primarily differed in that the quantum model allows for constructive judgments. Because the empirical findings showed an impact from the number of intermediate judgments, the proposed classical model performed poorly.

Kvam et al. (2015) compared a classical random walk drift diffusion model and a corresponding quantum version. Drift diffusion models provide the predominant approach to evidence accumulation (Ratcliff & Smith, 2004) and, as classical models, assume knowledge of the relative evidence for the available hypotheses at each time point exists (i.e., any uncertainty is epistemic). In such models, there is no default mechanism for how an intermediate judgment affects performance. Kvam et al. (2015) noted that both the classical and the quantum models can be informed by intermediate choices. However, in the former case, the choice does not change the evidence/ state, only the information that may be available to the decision maker. A Bayesian model comparison favored the quantum model. A technical issue is that the initial states for the quantum and classical models were not matched. If both models involved an initial Gaussian state, then both classical and quantum evolution would result in a state conforming to a Gaussian distribution. Then, any final state difference could be attributed to the measurement. Also, one could question how applicable the Markov random walk model was in this case, since accumulator type models are more typical in perceptual decision making tasks, unlike Kvam et al.'s (2015) approach. For example, Markov random walk models have no mechanism for arriving at a decision without an external prompt, but stimulus presentation in Kvam et al.'s (2015) paradigm was sometimes very long, suggesting that stopping behavior would be internally driven.

5.7 Decision order effects

5.7.1 Empirical research

There is abundant evidence for order effects in decision making. For example, in a well-known investigation based on Gallup polls, the probability to answer yes to a question about whether Clinton is trustworthy depended on whether an analogous question about Gore preceded or followed the Clinton one (Moore, 2002). There are similar results in medical diagnosis, whereby the assessment for the probability of a disease depended on the order of conditionalizing evidence, even for medical trainees (Bergus et al., 1998), and in a jury decision task (McKenzie, Lee, & Chen, 2002; Trueblood & Busemeyer, 2011). A related finding concerns differences in the evaluation of public service announcements, whether from the perspective of the self vs. the perspective of another observer (Wang & Busemeyer, 2015). Question order effects can be identified as primacy vs. recency effects or contrast vs. assimilation effects, referring to the various ways questions or pieces of evidence interact/ combine (Hogarth & Einhorn, 1992; Payne et al., 1993; Wang & Busemeyer, 2013).

Is it possible to establish a relation between the pattern for yes and no responses, for two binary questions, presented in all possible orders? Abbreviating $A = yes$ to A_{yes} etc., consider the following quantity:
$\left[Prob\left(A_{yes}, B_{yes}\right) + Prob(A_{no}, B_{no})\right] - \left[Prob\left(B_{yes}, A_{yes}\right) + Prob(B_{no}, A_{no})\right] = \left[Prob\left(A_{yes}, B_{no}\right) + Prob\left(A_{no}, B_{yes}\right)\right] - \left[Prob\left(B_{yes}, A_{no}\right) + Prob\left(B_{no}, A_{yes}\right)\right]$. Wang et al. (2014; Wang & Busemeyer, 2013; Yearsley & Busemeyer, 2016) proposed that this quantity is zero and called the resulting equality the quantum question (QQ) equality. Wang et al. (2014; Wang & Busemeyer, 2013) examined the QQ equality across 70 surveys, with participants varying between 651 and 3,006, and reported consistency

with the QQ equality in nearly all cases. Yearsley and Trueblood (2018) considered a variant based on conditional probabilities, $Prob(N|\bar{B}, A) - Prob(N|A, \bar{B}) = Prob(N|\bar{A}, B) - Prob(N|B, \bar{A}) = 0$, and provided supporting empirical evidence.

Yearsley and Trueblood (2018) provided a novel test of the co-occurrence between order effects and conjunction fallacies. The authors collected judgments regarding five main candidates for the Republican and Democratic 2016 presidential nominations, concerning the probability of each candidate winning various primaries and the eventual nomination. Order effects in the conditionalizing information were observed ($Prob(N|\bar{B}, A) \neq Prob(N|A, \bar{B})$) and conjunction fallacies ($Prob(A \wedge B) > \min\{Prob(A), Prob(B)\}$), but no double conjunction fallacies. Importantly, both effects were observed for the majority of participants. Note, as for Trueblood et al. (2017), a more analytic style of thinking was more likely to be associated with compatible representations.

Finally, the idea that previous questions can provide a unique context or perspective for subsequent ones (e.g., Schwarz, 2007) was considered using the so-called ABA paradigm (Khrennikov, Basieva, Dzhafarov, & Busemeyer, 2014). Suppose that order effects are identified for questions A, B in an initial experiment. Then in another experiment, question A is presented, followed by B, followed by A again. Khrennikov et al. (2014) suggested that naïve observers would aim to be consistent across the copies of A, regardless of the presence of B in between or the relation between the A, B questions. Busemeyer and Wang (2017) conducted an ABA experiment with 325 participants on sets of questions for which order effects had been previously identified and reported a moderate amount of opinion change between the first and second iteration of the A question.

5.7.2 Quantum cognitive models

Regarding question order effects, Trueblood and Busemeyer's (2011) approach encompassed Bergus et al.'s (1998) medical diagnosis task, McKenzie et al.'s (2002) jury decision task from, and new experiments. They specified a mental state for the possible combinations between the presence, absence of e.g. a disease and positive or negative evidence, in a composite (direct sum) space, such that one subspace corresponded to the disease being present and the other absent. The initial story and subsequent pieces of evidence were modeled with different rotations of this mental state. The specification of the Hamiltonians was broadly analogous to that of Pothos and Busemeyer (2009) for the disjunction effect (Shafir & Tversky, 1992), that is, there was a part that just operated within each subspace and a part that mixed amplitudes across the two subspaces and could lead to violations of the law of the total probability. Failures of commutativity in conditionalizing conjunctions (e.g., $Prob(disease|evidence1 \ \&\ evidence2)$) in their model arose because Hamiltonians did not commute with the projectors for intermediate judgments about the presence of the disease given the presented evidence at each step.

An alternative approach to question order effects was that of Wang and Busemeyer (2015; see also Wang & Busemeyer, 2016a), who employed a quantum random walk model for modelling the effectiveness of public health service announcements from the perspective of the self and then another person in one order vs. the reverse order. The model was analogous to Kvam et al.'s (2015).

The QQ equality is an a priori prediction of quantum theory in decision making. It can be derived assuming that no new information is included between questions. In two dimensions, the QQ equality relates to the way the positive and negative answers for two questions (two different basis sets in

Hilbert space) are linked up – the unitary transform that takes us from the basis set for one question to that of another is fixed by the relation between any one answer to one question and another to the other question. Yearsley and Trueblood's (2018) QQ equality variant requires one-dimensional subspaces and reciprocity (reciprocity is automatically true when A, B are represented by one-dimensional subspaces).

Yearsley and Trueblood's (2018) main contribution was to identify incompatibility as the driving factor for both order effects and conjunction fallacies and so predict a co-occurrence for these effects. They presented a simple quantum model with one-dimensional subspaces for the effects of interest and extracted constraints on the size of the effects.

The ABA paradigm was proposed as a test of how incompatibility can moderate our expectation of question order effects. The correlation between the two responses of A should reveal identity when A, B are compatible, but it would be otherwise bounded depending on the uncertainty relation between A, B (Khrennikov et al., 2014). That is, if a response to question A is followed by an incompatible question B, then the answer to question A has to become uncertain again and the correlation between the two A copies should be less than 1. These expectations were refined with Wang and Busemeyer's (2016) quantum model for ratings, which showed that the squared correlation between the first and second A measurements should be positively related to the squared correlation between the A, B measurements.

5.7.3 Critical evaluation and controversy

Trueblood and Busemeyer (2011) reported excellent fits of the quantum model with experimental results. It is appealing that the same quantum formalism was employed for both order effects and the disjunction effect (cf. Pothos & Busemeyer, 2009).

Question order effects are classically puzzling, because of commutativity in conjunction. Suppose A, B represent the answer yes to corresponding questions. Then, the event answering yes to the first question and then yes to the second one is given by $Prob(A)Prob(B|A) = Prob(A\&B) = Prob(B)Prob(A|B)$ and so accommodating order effects can only be done by conditionalizing on e.g. order. As was the case for constructive influences in judgment, there are several baseline intuitions for how earlier judgments could affect later ones, e.g., earlier questions may create a unique context for later ones (Schwarz, 2007). Hogarth and Einhorn's (1992) anchoring and adjustment model can describe order effects, but McKenzie et al. (2002) argued that across reasonable parameterizations the model could not reproduce their results. McKenzie et al. (2002) proposed a variant called the minimum acceptable strength model. In the standard anchoring and adjustment model evidence for a hypothesis depends on the most recent evidence and the accumulated weight for the hypothesis. The impact of the most recent piece of evidence depends on a fixed reference point. McKenzie et al.'s (2002) extension involved a variable reference point, achieving better description, but at the expense of more parameters. Trueblood and Busemeyer (2011) compared this minimum acceptable strength model with the quantum model and still concluded in favor of the latter.

Wang and Busemeyer (2015) compared their quantum model with a closely matched Markov random walk model and reported superior fits for the quantum model for their empirical results. This was largely because in the quantum model the two questions were represented as incompatible, so that in consecutive judgments order/ interference effects can emerge; this was not possible in the Markov random walk model.

Overall, we see two quantum approaches (Trueblood & Busemeyer, 2011; Wang & Busemeyer, 2015) for question order effects, both based on dynamical evolution of a mental state, but involving different specification. Arguably, this can be justified through a focus on comparisons with particular non-quantum models. Note, violations of commutativity in quantum conjunctions can also be captured with non-dynamical approaches.

The QQ equality is considered one of the most impressive predictions from the quantum cognition research programme, because it is a parameter free, stringent a priori constraint. Kellen et al. (in press) presented a class of models allowing varying formats and degree for the dependence of later questions onto earlier ones. They identified versions which either produced the QQ equality exactly or showed the QQ equality as a likely prediction, thus contesting the necessity for the quantum formalism to account for the QQ equality. An advantage of Kellen et al.'s (in press) approach is that it enables a deeper process understanding of the relevant psychological mechanism, in that their various models are specified more directly in terms of psychological processes. By contrast, formal probabilistic models (including quantum ones) require further interpretational steps before process parts can be recast in traditional psychological terms. An advantage of the quantum approach is that the QQ equality can be predicted a priori, as well as the conditions under which non-conformity is expected. Overall, it is unsurprising that heuristic principles can be employed to reproduce the QQ equality. But, we would argue that an explanation based on heuristics and one on formal probability principles serve different purposes. Note, it is unclear whether Kellen et al.'s (in press) analysis can reproduce the QQ equality variant that Yearsley and Trueblood (2018) reported.

Yearsley and Trueblood (2018) reported the predicted co-occurrence of order effects with conjunction fallacies and consistency between empirical results and the size of these effects. They also observed a dependence of style of thinking (analytic vs. reflexive) and political identity on the likelihood of incompatible representations. Yearsley and Trueblood (2018) examined whether Hogarth and Einhorn's (1992) anchoring and adjustment model can reproduce their QQ equality variant, but this was not possible without implausible assumptions. They also considered Costello and Watts's (2014) probability theory plus noise model for the conjunction fallacy, but they derived a constraint from this theory not satisfied by their data. They noted that there are no existing non-quantum models which require a co-occurrence of order effects and conjunction fallacies or that predict a dependency of such effects on cognitive style of thinking.

The results of the ABA paradigm showed the squared correlation between the first and second A measurements to positively relate to the squared correlation between the AB measurements, as predicted by the quantum model (Busemeyer and Wang, 2017). Even though there is non-quantum psychological theory for how previous questions can alter the context for subsequent ones (e.g., Schwarz, 2007), the dependence of this effect on the compatibility of A, B, as well as the precise quantitative predictions, are beyond any such baseline intuitions. Classical test theory (e.g., Lord & Novick, 2013) also postulates variability in measurements across occasions. According to this view, the measurement result from e.g. a question rating represents both a true opinion that is fixed across repeated measurements plus some measurement error that randomly varies across repeated measurements. However, the measurement error model differs from the quantum approach in a fundamental manner. When considering an AA sequence (i.e., responding to A twice without B in between), the projection principle of quantum probability predicts perfect correlation between the first and second measurements of A, whereas classical test theory allows measurement error to occur across repetitions; empirical results support the quantum view (Busemeyer & Wang, 2017)[3]. Classical test theory also cannot predict the role of compatibility in whether there is identity between the two iterations of A in an ABA sequence.

5.8 Conjunction/ disjunction fallacies in decision making

5.8.1 Empirical research

There are a range of related results, including conjunction fallacies (Tversky & Kahneman, 1983; Moro, 2009), disjunction fallacies (Bar-Hillel & Neter, 1993; Carlson & Yates, 1989; Fisk, 2002), unpacking effects (Rottenstreich & Tversky, 1997; Sloman, Rottenstreich, Wisniewski, Hadjichristidis, & Fox, 2004), and more complex conjunctions (e.g., Winman, Nilsson, Juslin, & Hansson, 2010). Using the terminology from Tversky and Kahneman's (1983) paradigmatic Linda example, where BT=bank teller, F=feminist, the conjunction fallacy is $Prob(F \wedge \ BT) > Prob(BT)$. The disjunction fallacy is $Prob(F \vee \ BT) < Prob(F)$ (note, this is not to be confused with the disjunction effect, see shortly; Shafir & Tversky, 1992). Conjunction effects often arise when combining an unlikely conjunct with a likely one and/or when the conjuncts have a strong causal association. Unpacking effects are violations of the law of total probability e.g. instances of subadditivity, whereby $Prob(A) < Prob(A \wedge \ B) + Prob(A \wedge {\sim}B)$.

Gronchi and Strambini (2017) presented a variant of the Linda problem to test the Wigner–d'Espagnat inequality, $Prob(A \cap \bar{C}) \leq Prob\big((A \cap \bar{B}) \cup (B \cap \bar{C})\big)$ (d'Espagnat, 1979). The Wigner–d'Espagnat inequality is a version of Bell's inequality (Bell, 2004) better suited to Linda-style paradigms and a Venn diagram illustrates how trivial it is from a classical perspective (i.e., if all possibilities are represented in the same sample space). Bell notably stated "trivial as it is, the inequality is not respected by quantum mechanical probabilities'' (Bell, 1981, p. 52). Gronchi and Strambini (2017) reported two experiments concerning the probability of picking different objects from an urn or of male vs. female students more likely to be playing soccer. In the first experiment the results indicated violations of the Wigner-d'Espagnat inequality. In a second experiment, when participants directly estimated unpacked probabilities ($Prob(A \cap \bar{B})$ and $Prob(B \cap \bar{C})$ separately) Wigner–d'Espagnat violations disappeared, thus indicating that a subadditivity pattern ($Prob\big((A \cap \bar{B}) \cup (B \cap \bar{C})\big) \leq Prob(A \cap \bar{B}) + Prob(B \cap \bar{C})$) possibly accounts for Wigner-d'Espagnat violations.

The disjunction effect is an influential demonstration of a violation of the law of total probability (Shafir & Tversky, 1992; Tversky & Shafir, 1992). In one variant, participants were presented with a prisoner's dilemma game, such that the payoffs recommended defection on the assumption that the hypothetical other player cooperates (Section 5.6.1). Participants received different information about the intention of the other player: that he/she would cooperate, defect, or no information would be given. In each of the 'known' conditions, most participants preferred to defect, but in the 'unknown' condition many reversed their judgment to cooperate, revealing a pattern of

[3] Note using POVMs instead of projectors would predict non-identity for consecutive judgments. Currently there is a challenge in understanding the circumstances when POVMs vs. projectors are more suitable.

$Prob(defect; unknown) < Prob(defect|cooperate) \cdot Prob(cooperate) + Prob(defect|defect) \cdot Prob(defect)$, where the conditionalizations refer to the provided information ($Prob(defect; unknown)$ just refers to the marginal when there is no information about the other player's action). The marginal in the unknown condition has to be bounded by the conditionals in each of the known conditions, so that the results indicated a violation of the law of total probability. Other variants were proposed (e.g., a two stage gambling task) and the finding has been widely replicated. Also, extensions to the prisoner's dilemma task have been developed, e.g., a sequential version (Blanco et al., 2014) or one to study prior commitments to defect or cooperate (Kvam et al., 2014).

Another violation of the law of total probability was reported in Townsend et al. (2000; Busemeyer et al., 2009). Participants were first presented with a face. Some participants were asked to categorize the face as friendly or hostile, while other participants simply observed it. In a second step, all participants were asked to indicate an action, attack or withdraw. Across several such stimuli, we expect $Prob(attack) = Prob(attack \wedge friendly) + Prob(attack \wedge \sim friendly)$, but this was not observed. Wang and Busemeyer (2016b) replicated this categorization-decision effect.

5.8.2 Quantum cognitive models

Busemeyer et al. (2011) employed a static model in which a conjunction fallacy emerges, mostly on the basis of suitable representation assumptions. Specifically, $|P_{BT}|\psi\rangle|^2 = |P_{BT}P_F|\psi\rangle|^2 + |P_{BT}P_{\sim F}|\psi\rangle|^2 + \langle\psi_{BT,\sim F}|\psi_{B,F}\rangle + \langle\psi_{BT,F}|\psi_{BT,\sim F}\rangle$, where the interference term is $\Delta = \langle\psi_{BT,\sim F}|\psi_{BT,F}\rangle + \langle\psi_{BT,F}|\psi_{BT,\sim F}\rangle$ (Section 2). Note, this decomposition implies that the conjunctive statement F&BT is evaluated in terms of a projection first to the F subspace and then to the BT one, an assumption justified through appeal to heuristics prioritizing the processing of more likely information (Gigerenzer & Todd, 1996). If the interference term is negative, then the conjunction fallacy follows. A negative interference term obtains with two conditions. First, $|P_{BT}P_{\sim F}|\psi\rangle|^2$ has to be small, which means that the ~F property makes the BT one unlikely. In the Linda problem, we can ask: does knowledge that Linda is not a feminist make the BT property likely or unlikely? This is a reasonable intuition, since the knowledge that a person is not a feminist does not indicate a particular profession. Second, the interference term has to be negative, which is the case if the state produced from the sequence $P_{BT}P_{\sim F}\psi$ is opposite compared to $P_{BT}P_F\psi$. This is reasonable, since the characteristics of a feminist bank teller are plausibly opposite to those of a non-feminist bank teller.

Gronchi and Strambini (2017) produced two quantum models to account for violations of Wigner–d'Espagnat. One model is based on incompatible questions in the same space and so this approach is analogous to that of Busemeyer et al. (2011). The other model assumes that the two questions for each logical operation in the Wigner–d'Espagnat equation are evaluated in separate spaces and so they are compatible. Any violations of Wigner–d'Espagnat then emerge from entanglement. The two models were shown to be equivalent.

Pothos and Busemeyer (2009) proposed a dynamical model for the disjunction effect. Each possibility for the opponent's action (defect, cooperate) was modeled with a separate Hamiltonian/ unitary corresponding to how payoff affects probability to defect. Following a classical intuition, the dynamical process in the unknown case can be specified from that of the known cases as $U_D(t) \oplus U_C(t)$, so that

$Prob(D; unknown\ case) = |M_{D,D} \oplus M_{D,C} U_D(t) \oplus U_C(t)\psi_D \oplus \psi_C|^2 =$
$|M_{D,D}U_D(t)\psi_D \oplus M_{D,C}U_C(t)\psi_C|^2 = Prob(D\&D) + Prob(D\&C)$, where $M_{D,D}$ is a measurement operator (a projector) for the probability to defect, when the participant knows the opponent will defect. Thus, if the mental state changes by a dynamical process of the form $U_D(t) \oplus U_C(t)$ the law of total probability is obeyed. Quantum theory allows more complex dynamical processes and Pothos and Busemeyer (2009) employed a Hamiltonian with a form $H_D \oplus H_C + H_{Mixer}$ so that the corresponding unitary can no longer be written as like $U_D(t) \oplus U_C(t)$. Interference effects can then emerge, allowing for violations of the law of total probability. Psychologically, H_{Mixer} aligns beliefs and actions, thus reducing cognitive dissonance (Festinger, 1957). Denolf et al. (2017) proposed an analogous quantum model for a sequential version of the Prisoner's Dilemma game (from Blanco et al., 2014). The main features in their model were an elaboration of assumptions regarding whether actions or beliefs could be considered compatible or incompatible.

For the categorization, decision paradigm, Wang and Busemeyer (2016b) also employed a Hamiltonian with structure $H_{Friendly} \oplus H_{Hostile}$, corresponding to how knowledge of each possible categorization can affect decision, which was extended to $H_{Friendly} \oplus H_{Hostile} + H_{mixer}$, to allow interference effects and violations of the law of total probability; the mixer aligns categorization with decision. Moreira and Wichert (2016) adopted an alternative approach, employing Tucci's (1995) quantum-like network theory, which propagates amplitudes instead of probabilities. Because amplitudes propagate across multiple paths, quantum nets can give rise to interference effects, not possible in Bayesian networks. Moreira and Wichert (2016) computed the interference terms in their quantum network from the face images in the categorization, decision paradigm (from Busemeyer et al., 2009). They converted the images into vectors, stochastically determined assignment into good and bad faces, and computed cosine similarities between images in the different good/bad categories. These similarities were employed as interference terms in the quantum network model.

Yukalov and Sornette (2011) provided an alternative quantum framework primarily aimed at the disjunction effect and conjunction fallacy. They defined a set of prospects, treated as quantum theory operators, which lead to a quantity analogous to classical expected utility. This classical expected utility quantity is complemented with a non-classical one, called an attraction factor, which arises from interference effects in composite prospects. Large attraction factors are assumed to result in more attractive prospects. For the disjunction effect, there is an attraction term for each action, analogous to quantum interference terms. Using a uniformity assumption for some of the variables, uniform priors, and an uncertainty aversion principle (reluctance to act under larger uncertainty), they reconstructed probabilities consistent with the disjunction effect. Regarding the conjunction fallacy, they used the quantum law of total probability (Busemeyer et al., 2011) and involved the uncertainty principle to justify a negative inference term.

5.8.3 Critical evaluation and controversy

Busemeyer et al.'s (2011) quantum model arguably revolutionarizes our notion of correctness in probabilistic inference, which has been dominated by classical probability theory to the point that it is hard to envisage alternatives. With probabilities as subsets of some sample space, it is impossible to imagine a conjunction as more likely than a marginal, yet the geometric picture of probabilities in quantum theory allows an intuition of how this can be so. Busemeyer et al.'s (2011) work encompassed several related fallacies (the disjunction fallacy, unpacking effects, more complex conjunctions etc.). If the conjunction fallacy can be 'correct', with quantum probabilities, can it also be rational? Pothos et al. (2017) showed that quantum theory is consistent with the Dutch Book criterion for rational behavior (de Finetti et al., 1993) and that a conjunction fallacy can be rational when the questions are contextual (the presence of one question alters the meaning of the other) or disturbing.

In Busemeyer et al.'s (2011) model a conjunction fallacy arises from incompatibility. Therefore, manipulations that encourage processing of the conjuncts in a concurrent (compatible) way should reduce conjunction fallacies. There is such evidence from training on the algebra of sets (Agnoli & Krantz, 1989; see also Yamagishi, 2003, and Wolfe & Reyna, 2010) to feedback training on the classical probability rule (Nilsson, 2008). Familiarity is also thought to induce transitions from incompatible to compatible representations and Nilsson et al. (2013) reported some evidence for the conjunction fallacy (Nilsson et al., 2013). Recall also Yearsley and Trueblood (in press) who demonstrated the co-occurrence of conjunction fallacies and order effects. Busemeyer et al. (2011) assumed that regardless of the presentation order of the two conjuncts, the more likely one is processed first. There is some evidence that manipulating processing order (e.g., through priming) supports the quantum account (Stolarz-Fantino et al., 2003, Experiment 2; Gavanski & Roskos-Ewoldsen, 1991). A critical observation concerns the definition of disjunction in the quantum account, which does not benefit from as clear a justification as the sequential conjunction.

There have been several models for the conjunction fallacy. Tversky and Kahneman's (1983; Shafir et al., 1990) proposed a representativeness heuristic, so that probability judgments are essentially similarity processes. Thus, probabilistic inference is relegated to a more generic cognitive process. However, the similarity mechanism in representativeness is underspecified. Because the quantum model computes probability as representational overlap in a multidimensional space, it has been advocated as a particular expression of probabilistic inference as similarity (cf. Sloman, 1993). Averaging models compute conjunctions as averages of the probabilities of the conjuncts (Abelson, Leddo, & Gross, 1987; Fantino, Kulik, & Stolarz-Fantino, 1997; Nilsson, 2008; Nilsson et al., 2009). These accounts predict conjunction fallacies regardless of causal links between the conjuncts (high rate of conjunction fallacy) or not (lower rate). However, we expect a conjunction fallacy in cases such as "Mr. F has had one or more heart attacks and Mr. F is older than 55" but not in "Mr. F has had one or more heart attacks and Mr. G is older than 55" (Tversky and Kahneman, 1983).

According to the inductive confirmation account, probability estimates are driven by the difference $Prob(h|e) - Prob(h)$, where *h* is a hypothesis and *e* is evidence (Tentori et al., 2013). This account dissociates the degree of conjunction fallacy from the probability of the more likely conjunct. Tentori et al.'s (2013) task involved a hypothetical woman who is Russian (R), lives in New York (NY), and in addition is an interpreter (I). Their main finding was $Prob(I \wedge NY|R) > Prob(NY|R)$, even though the *I* possibility is less likely than the ~*I* one (Tentori et al., 2013, express this as $Prob(\sim I|NY \wedge R) > Prob(I|NY \wedge R)$). Thus, in this scenario and variants the conjunction fallacy arises from a less likely conjunct (being an *I*). The quantum conjunction fallacy model can cover Tentori et al.'s (2013) results (Busemeyer et al., 2015). Additionally, Busemeyer et al. (2015) argued that order effects in the consideration of the conjuncts (Stolarz-Fantino et al., 2003, Experiment 2; Gavanski & Roskos-Ewoldsen, 1991) support the quantum account, but not the inductive confirmation one. Finally, the quantum account can accommodate manipulations that arguably encourage compatible representations (Agnoli & Krantz, 1989; Nilsson, 2008; Wolfe & Reyna, 2009; Yamagishi, 2003), including increasing familiarity (Nilsson et al., 2013), but consistency with the inductive confirmation proposal is unclear.

In the probability theory plus noise account decision making is based on classical probability theory, but probability estimates are subject to error (Costello & Watts, 2014). This error arises because probability estimates are based either on enumerating relevant memory instances or a mental simulation of such instances. Costello and Watts (in press) derived several probabilistic expressions which diverge depending on whether one adopts quantum rules or probability plus noise rules. Their results appear to uniformly support their model, not quantum theory. Note, quantum theorists have been incorporating noise in their work too (e.g., Trueblood et al., 2017; Yearsley and Pothos, 2016). However, noise has not been explored for the conjunction fallacy, because in quantum theory noise emerges as a small probability that the internal computation may produce A, but one may mistakenly pronounce ~A. Therefore, in quantum models the role of noise is relevant in situations of multiple decisions.

There are concerns for the noise model. First, the mechanism assumed to produce the noise is based on memory enumeration or mental simulation, but especially the latter seems implausible, for unfamiliar questions. Second, sampling independence for estimating related probabilities (e.g., marginals and a conjunction) is implausible, but without sampling independence the noise model cannot predict the conjunction fallacy. Third, Costello and Watts (2016) produced a conditional probability noise formula, but in a more recent paper (Costello & Watts, 2018), they eschewed their own conditional probability formula to model order effects using a memory priming parameter. Fourth, error is implausible for completely impossible or certain events. Fifth, Costello and Watts's (2014) prediction for a conjunction fallacy relies on P(A∧B) being close to P(A), but Tentori et al. (2013) reported conjunction fallacy results at odds with this requirement. Finally, the noise model cannot accommodate conjunction fallacies, when $P(A \wedge B) \sim P(A) > 0.5$; Yearsley and Trueblood (in press) reported such results.

Do Costello et al.'s (in press) results challenge the quantum model? Costello et al. (in press) consider familiarity to equate with compatibility and there is evidence that this is the case (e.g., Trueblood et al., 2017; Yearsley & Trueblood, in press). However, incompatibility may exist for familiar questions too, where questions may induce different perspectives to each other, as seems apparent for order effects regarding the Clinton, Gore honesty questions (Wang & Busemeyer, 2013). However, detailed coverage of the Costello et al. (in press) results from the quantum model is still needed.

Boyer-Kassem et al. (2016) argued that a quantum model for the conjunction fallacy entails three empirical predictions: (1) consistency with the so-called grand reciprocity equalities (which they derived), (2) the existence of order effects, and (3) consistency with the QQ equality. They reported results which they claimed go against the quantum model. Regarding the grand reciprocity equalities, these were derived assuming reciprocity and so one-dimensional representations, but Busemeyer et al.'s (2011) model is not restricted in this way and it seems unlikely that complex psychological concepts (like feminism) would have one-dimensional representations (Pothos et al., 2013).

Regarding the co-occurrence of order effects and conjunction fallacies, Boyer et al. (2016) considered questions from seven different scenarios, from well-known conjunction fallacy demonstrations, but reported order effects in only two cases, no order effects in two other cases, and ambiguous results in the remaining three cases. Note, they corrected for family-wise error using the Bonferroni correction, which has been criticized as conservative (Nakagawa, 2004; Perneger, 1998). Importantly, they did not demonstrate concurrent conjunction fallacies, as Yearsley and Trueblood (in press) did. It is possible that their sample was less prone to order effects and conjunction fallacies (their participants were students in economics, management, or medicine). Their report of failures to satisfy the QQ equality awaits further investigation.

Gronchi and Strambini's (2017) work brings into focus a key issue, that violations of the Wigner-d'Espagnat inequality can arise either because of incompatibility in the same space or compatible representations in separate spaces with an entangled state. Both incompatibility and entanglement preclude a joint probability distribution, but in the former case because of uncertainty relations and in the latter because of the impossibility to describe the corresponding entities independently of each other. However, researchers may require a commitment to incompatibility or compatibility in any situation. In Gronchi and Strambini's (2017) experiments, questions regarding the color or shape of objects are likely to be treated as compatible (because perceptual physical properties like this are unlikely to have a contextual influence on each other), but gender and likelihood of playing different sports incompatible (if different genders change our perspective for likely sports).

Pothos and Busemeyer (2009) showed that the quantum model could describe the disjunction effect better than a matched classical probability model (employing the Kolmogorov forward equation, which is the classical equivalent of Schrodinger's equation). In Pothos and Busemeyer's (2009) approach the transition from classical (no interference) to quantum (interference possible) probabilities occurs within the same framework. Arguably, this model (first described in Busemeyer et al., 2006) was the first quantum cognitive model based on a simple application of quantum probabilities, presented in a journal followed by psychologists. This framework has been employed for other broadly related findings (Trueblood & Busemeyer, 2011; Wang & Busemeyer, 2016). Tversky and Shafir's (1992) explanation for the disjunction effect was based on the idea of failure of consequential reasoning. In prisoner's dilemma participants may have a good reason to defect knowing the opponent defects and a good (different) reason knowing the opponent cooperates, but lack clarity of why they should defect in the unknown case. Clearly, such an explanation loosely constrains specific technical approaches. One criticism of Pothos and Busemeyer's (2009) model is that, because of its use of unitary dynamics, probabilities are perpetually oscillating and Pothos and Busemeyer (2009) assumed a cutoff for decisions. The issue of stabilization in quantum dynamics is formally addressed with open system dynamics (e.g., Asano et al., 2011a, 2011b).

Similar considerations apply to Wang and Busemeyer's (2016b) quantum model for categorization, decision results. Additionally, these authors discussed a signal detection theory model (Ashby & Townsend, 1986), for which, following categorization, the eventual decision can still depend on the stimulus; for the Markov and quantum models, once a categorization is made, the decision depends only on the categorization. The authors argued that signal detection theory would have trouble accounting for interference, unless one introduces post hoc assumptions regarding how the response regions (for the categorizations and/or the decisions) change boundaries depending on stimulus categorization.

Denolf et al. (2017) allowed outcomes of a rating scale to be non-orthogonal vectors, which is unorthodox in quantum theory. However, their approach is broadly equivalent to an assumption of

POVM (as opposed to projective) measurement. Its advantage is that it enables lower dimensionality spaces and non-repeatability of measurements.

Moreira and Wichert's (2017) quantum model for the categorization, decision violation of the law of total probability raises some questions. It is unclear that the vectors in the quantum network are equivalent to the ones employed to compute the similarities. That is, there is some doubt in whether the equivalence between similarities and quantum interference terms is valid. Additionally, Moreira and Wichert (2017) assumed a renormalization of the image vectors to occupy the negative part of the vector space, because this leads to a restriction to just destructive interference effects (as empirically observed), but this assumption could do with further justification.

Yukalov and Sornette's (2011) framework was presented as a general framework for probabilistic decision making, with applications relating to the disjunction effect and the conjunction fallacy. However, there are some weaknesses. Regarding the disjunction effect, the uncertainty aversion principle is translated to a condition that the uncertainty factor for acting is positive and the for not acting negative, an assumption of questionable usefulness, because in some paradigms it is unclear which option corresponds to acting or not acting. For example, in Tversky and Shafir's (1992) prisoner's dilemma, there is a suppression of the probability to defect in the unknown condition and an increase in the probability to cooperate. But how can uncertainty aversion inform a higher propensity to cooperate? Regarding the conjunction fallacy, it is also unclear how uncertainty aversion operates, even though this ostensibly determines the right sign for the interference term. That is, in what way could one assume that the probability of a marginal is more uncertain than the probability of conjunctions, since normatively the law of total probability requires equality?

5.9 Other judgment phenomena

5.9.1 Empirical research

Basieva et al. (2017) considered probabilistic updating. The authors designed a task based on a hypothetical crime mystery, with several potential suspects. Participants were initially presented with some information for guilt for the various suspects and then with more information, which was meant to overturn expectations about initially unlikely suspects.

Aerts et al. (2018) sought to provide a demonstration of entanglement in decision making. Participants were asked to pick a pair of wind directions (e.g., North and Southwest vs. South and Northeast), that they considered 'good examples of two different wind directions'. Across several trials, participants were presented with choice sets including different combinations of pairs of wind directions. By analogy with entanglement experiments in physics, a first system would involve two directions A, A' and a second system two different wind directions B, B', so that overall we would have four combinations of wind directions (AB, AB' etc.), with each combination specifying a trial (since e.g. each of A, B are binary questions, there are four possibilities for the combined question A, B and participants would choose one). The design was fully within participants. The main empirical finding was that averaged participant responses violated the CHSH inequality, so that a complete joint probability distribution for the relevant events is impossible.

The Ellsberg paradox (Ellsberg, 1961) is a decision task involving drawing colored balls from an urn. The task is to decide which color will be most likely. A particular color is associated with a fixed probability (say red), but another one with an unknown probability (say green). When participants are told to choose between a bet on drawing a red ball vs. a green one, then they choose the former. When participants are told to choose between a bet on not drawing a red ball vs. not a green one, then they also choose the former, displaying ambiguity aversion (aversion to ambiguous probabilities) and violating expected utility theory. Dimmock et al. (2015) showed that there is ambiguity aversion for medium and high probabilities but ambiguity seeking for low probabilities. The Allais paradox (Allais, 1953) involves a choice between two gambles A, B and a second choice between two further ones C, D. Even though A and C are equivalent, in the first choice A is preferred but in the second C is not preferred, violating the independence axiom of expected utility theory.

5.9.2 Quantum cognitive models

Basieva et al. (2017) compared classical probability updating (based on Bayes law) with quantum probability updating (based on Luder's law). Luder's law can be expressed as $Prob(A|B) = \frac{|P_A P_B|\psi\rangle|^2}{|P_B|\psi\rangle|^2} = \frac{Prob(B \wedge then\ A)}{Prob(B)}$ and is equivalent to Bayesian probabilistic updating, the difference being that classical conjunction is replaced by the quantum equivalent for incompatible questions, which is sequential conjunction.

Aerts et al. (2018) represented the questions about wind directions with tensor products of the form $(\sigma \cdot a) \otimes (\sigma \cdot b)$, where *a*, *b* are direction vectors and $\sigma = \sigma_x + \sigma_y + \sigma_z$, the Pauli spin matrices. In order to describe correlations between binary measurements in two subsystems (i.e., the responses for each of two wind directions, for each subsystem), one needs to identify four direction vectors *a*, *b*, *a*', *b*' and an entangled state. The authors claim that there is no guarantee of a suitable quantum representation of this kind. But, they identified an entangled mental state and wind directions which described the observed correlations.

LaMura (2009) provided a quantum framework for expected utility, called projective expected utility theory. Whereas expected utility theory is based on objective outcomes, projective expected utility is based on subjective consequences, which are obtained as a basis change from objective outcomes. The use of a quantum framework for obtaining probabilities allows for interference effects and contextual dependence, which can accommodate the Ellsberg and Allais paradoxes, amongst other results. Al Nowaiti and Dhami (2017) focused on the Ellsberg paradox, aiming to capture both ambiguity aversion and ambiguity seeking (Dimmock et al., 2015). They employed Feynman's rules for state transitions, when the intermediate steps are known vs. not known. Briefly, in a transition $\varphi \rightarrow \psi$ through χ_1, χ_2, when the intermediate state is observed, then the overall transition probabilities are obtained by adding probabilities. However, when it is not observed, then the overall transition probabilities are obtained by adding amplitudes, which can lead to interference effects. This approach could accommodate Dimmock et al.'s (2015) findings, depending on assumptions of whether cognitive processing is overt (probabilities combine) or covert (amplitudes would combine), without parameter manipulation. Ellsberg and Allais have been the focus of other proposals (Aerts et al., 2014; Busemeyer & Bruza, 2012; Khrennikov & Haven, 2009), which all share some key elements (the use of interference terms/ contextuality to accommodate the paradoxes).

5.8.3 Critical evaluation and controversy

Basieva et al. (2017) argued that Bayes rule provides incomplete coverage of everyday inference, because it precludes large jumps from priors to posteriors. Classically updated probabilities are linearly dependent on priors and the ratio of likelihoods is constrained by the ratio of priors, so that low priors will restrict posteriors. Additionally, zero priors require zero posteriors, a situation called by the authors the zero priors trap. The zero priors trap has a long history. Oliver Cromwell allegedly said to the members of the synod of the Church of Scotland "to think it possible that you may be mistaken" (Carlyle, 1885). The idea here is that a small probability should be assigned to even highly improbable situations. Shafer's (1975) model can circumvent the zero priors trap in an otherwise Bayesian framework, by organizing hypotheses into groups and assigning a prior probability to each group. Crucially, there is flexibility in how group probability is divided amongst individual hypotheses, and new information could lead to a reassignment that means that we no longer need to update from a zero prior. However, Basieva et al. (2017) showed that Shafer's (1975) hypothesis cannot explain their results, but Luder's law can.

Aerts et al. (2018) partly motivated their work as a test of CHSH violation consistent with the condition for no-signaling from Dzhafarov and Kujala (2014; Section 5.4.2). Their baseline data still violated this condition, but a transformation produced CHSH violation together with consistency to the no-signaling condition. Dzhafarov et al. (2016) questioned this transformation and argued that signaling accounts for CHSH violations in Aerts et al.'s (2018) experiments.

Regarding coverage of Ellsberg and Allais paradoxes, quantum applications (Aerts et al., 2014; Al Nowaiti and Dhami, 2017; Busemeyer & Bruza, 2012; Khrennikov & Haven, 2009; LaMura, 2009) share the use of interference and contextuality. Al Nowaiti and Dhami (2017) extracted probabilities consistent with the Ellsberg paradox, without parameter manipulation (contrast with Busemeyer & Bruza, 2012; LaMura, 2009). A question for such approaches is their capacity for generative value. Note also that detailed comparisons with non-quantum expected utility alternatives have not been carried out. For example, the Ellsberg paradox has also been approached through probability transformations (e.g., Klibanoff et al., 2005).

6. Is it worth persevering with quantum cognition models?

Having reviewed several cognitive applications of quantum theory, we next address questions critical for this research programme. Is it worth persevering with quantum models? Inevitably we are led to fundamental questions like what constitutes a strong theoretical framework and a good model.

A healthy theoretical framework arguably fulfills three conditions. First, its principles are interconnected, as this limits arbitrariness. Second, there are clear narratives about psychological process. Third, there are plausible assumptions regarding computational demands for the brain. We think the prospects regarding quantum theory are good. The first characteristic is self-evident in that quantum theory, as a formal probabilistic system, is axiomatically defined. In any specific quantum model some additional assumptions might be needed, as a way to bridge an abstract mathematical framework and cognition. However, in most cognitive quantum models such assumptions have been limited (cf. Jones & Love, 2011). Secondly, the elements of quantum models are typically assigned psychological meaning. For example, in static models, representations are typically set by reference to the empirical situation and projection sequences justified as embodying relevant contextual influences

(Busemeyer et al., 2011; Pothos et al., 2013; Wang et al., 2014). Entangled states are interpreted as instances of strong connectedness between the relevant entities (e.g., Bruza et al., 2015). In dynamical models, Hamiltonian parameters typically indicate drifts to the various options and often interpreted as relevant utilities (e.g., Pothos & Busemeyer, 2009; Trueblood & Busemeyer, 2011).

Finally, the problem of computational tractability is shared by many modelling frameworks. Focusing on quantum vs. classical probabilities, the former are often simpler than the latter, because incompatibility can reduce the complexity of the required joint probability distributions. To use Tversky and Kahneman's (1983) Linda example, a classical probability approach requires conjunctions not just involving whether Linda is a bank teller or a feminist, but also where she lives, her relationships, how she looks etc. Even if many of these probabilities can be automatically set through suitable priors, the complexity of the probability distributions may still be prohibitive. We think the situation regarding brain tractability of the postulated operations in quantum theory is at least equivalent to or better than that for classical theory (note, heuristics simplifying classical computations could be adapted for use in quantum theory, e.g., Sanborn et al., 2010). Note, the idea that quantum probabilities may be simpler than classical ones exactly underwrites the point that they may be more likely to be adopted when decision makers are less engaged or familiar with a task (Trueblood et al., 2017; Yearsley & Trueblood, in press).

Regarding specific models within a framework, arguably success depends on balancing parsimony with empirical coverage, comparisons with other formalisms, and generative value. For the first criterion, we distinguish between models proposed by researchers from the psychology community (e.g., Busemeyer, Dzhafarov, Pothos, Trueblood, Wang, Yearsley) and ones from other research communities (physics and economics; e.g., Aerts, Atmanspacher, Haven, Khrennikov, Sozzo). Amongst psychologists, we think there is general agreement regarding the criteria for good models, as above, but in other disciplines such criteria differ. So, we focus this discussion on models proposed by psychology researchers and contest that most such models satisfy the above criteria. First, there is no evidence that quantum cognitive models are more complex than non-quantum ones. Where detailed complexity comparisons were carried out, they favored the quantum models (Busemeyer et al., 2015; Trueblood et al., 2017).

Second, quantum cognitive models have been examined against relevant non-quantum models. Regarding probabilistic fallacies, such as the disjunction effect (Shafir & Tversky, 1992) or the categorization decision paradigm (Townsend et al., 2000), quantum models have been compared with closely matched classical probability models (dynamical models based on the Kolmogorov forward equation, e.g., Pothos & Busemeyer, 2009; Trueblood & Busemeyer, 2011; Wang & Busemeyer, 2015). In such cases, we think the motivation for employing quantum probability is compelling: there are quantum and classical matched probabilistic approaches, human behavior indicates inconsistency with the latter because of apparent interference effects, so we adopt the former. In other cases, for example, similarity judgments, there has been a perception of adequate existing models (e.g., Ashby & Perrin, 1988; Krumhansl, 1988) but closer scrutiny revealed inconsistencies and other problems (Pothos et al., 2013; Yearsley et al., 2017). Note, one can ask whether there are results indicating a necessity for quantum probabilities. Inconsistencies with classical principles can typically be classically accommodated through conditionalization (Section 5.1.3; Dzhafarov & Kon, in press). However, quantum theory has been designed for such results and can often offer more elegant and parsimonious models.

The conjunction fallacy (Tversky & Kahneman, 1983) has attracted an enormous amount of attention and it is here that the quantum model has been scrutinized the most. Its main weaknesses concern the assumption that the predicates in the conjunction are ordered in a certain way and the origin of the assumption of incompatibility. We believe that its main strengths are wide coverage of related fallacies and the fact that it provides an alternative perspective for correctness and rationality in probabilistic inference (Pothos et al., 2017). We argued that alternative formalisms can suffer from narrow scope (Busemeyer et al., 2015) and incoherent assumptions (as for Costello & Watts, 2016).

One related question is this: are there results disproving quantum cognitive models? Boyer-Kassem et al. (2016) and Costello et al. (in press) argued that this the case, in terms of failures of consistency with the QQ equality, constraints that ought to be adhered to if people were employing quantum probabilities, and lack of association between different results predicted from incompatibility (e.g., order effects and conjunction fallacies). There are two responses. First, we argued that some of these criticisms are incomplete (Section 5.8.3). Second, we think it would be extremely surprising if all behavioral phenomena can be modeled within the quantum framework. In the same way that increasingly researchers are advocating a view of cognition as involving a Bayesian part and a non-Bayesian part (e.g., Kahneman, 2001; Sloman, 1996), we think there will be bounds of applicability in quantum cognition models. For example, regarding super-correlations, while quantum representations can violate the Bell bound, they are restricted to an alternative bound – the so-called Tsirelson bound – lending themselves to simple tests. Inevitably, certain cognitive processes would be outside the remit of probabilistic frameworks, classical or quantum and would require (plausibly) heuristic explanations. The question is then whether there are enough successful cognitive applications to warrant further interest in quantum models, and our answer is yes.

Regarding generative value, many of the major applications of quantum theory to cognition have involved established results, such as the conjunction fallacy (Tversky & Kahneman, 1983), and so a concern is whether the applicability of quantum models might be restricted to re-descriptions of previous results. However, this review has presented examples of quantum models which have had generative value in psychology. First, there have been examinations of compositionality in memory or conceptual combination, using the Bell/ CHSH inequality, providing new empirical tests and evidence (e.g., Aerts, 2009; Aerts et al., 2015; Bruza et al., 2015; Cervantes et al., in press). Second, even though constructive influences in psychology have been explored before (e.g., Sharot et al., 2010), quantum cognitive models make specific and constrained predictions regarding such influences, because of the requirement of the collapse of the state vector. Such predictions have been utilized in empirical demonstrations (Kvam et al., 2014; Yearsley & Pothos, 2016) and have been the basis for a proposal of a new decision bias (White et al., 2014, 2015). Third, the QQ equality has been a surprising, a priori prediction from quantum theory, supported in many data sets (Wang et al., 2014). Fourth, the links between familiarity, cognitive style of thinking (Frederick, 2005), and the emergence of fallacies, such as the conjunction fallacy and order effects (Trueblood et al., 2017; Yearsley & Trueblood, in press), have provided a novel individual differences perspective to dual route ideas in cognition. There are other predictions which have yet to merit detailed empirical examination, for example regarding sequencing in temporal structure (e.g., Atmanspacher & Filk, 2010). Overall, we think that the application of quantum theory to cognition has had considerable empirical generative impact.

Regarding generative promise, we finally note that quantum systems can allow speedier computations than classical systems, in certain cases, due to entanglement and superposition (e.g., Grover, 1997; Shor, 1994). Thus, the possibility arises that quantum-like computation at the cognitive level may be the factor partly enabling the apparent speed with which we are able to make inferences. Such ideas are potentially promising, but currently underdeveloped. One difficulty is that quantum vs. classical speedup has been documented only for specific problems and there is a challenge to translate them to psychological ones. Another is that quantum speedup typically requires quantum physical systems, for example it depends on quantum gates, which operate on (real) superposition or entangled states (e.g., Behrman et al., 2000; Dong et al., 2008). As we are committed to a classical brain, it is unclear how to apply such ideas to quantum cognition. An alternative way to approach this issue is whether probabilistic learning might be aided via quantum instead of classical representations and there are indications that this might be the case (Bond, 2018).

7. What are the main weaknesses of quantum cognitive models?

We have created this review with a critical mindset and we summarize some key points here. First, most quantum cognitive models depend on incompatibility, as incompatibility allows the interference effects necessary to violate the law of total probability, produce order effects etc. How can incompatibility be determined a priori? Some investigators assess incompatibility empirically, e.g., with order effects (Boyer-Kassem et al., 2016; Busemeyer & Wang, 2014; Yearsley & Trueblood, in press). Others have employed proxy measures for incompatibility, notably style of thinking (a more reflexive vs. reflective style of thinking is more likely to lead to incompatible representations; Frederick, 2005; Trueblood et al., 2017; Yearsley & Trueblood, in press). Low familiarity with question combinations is also thought to make incompatible representations more likely (Busemeyer et al., 2011; Costello et al., in press; Trueblood et al., 2017). Overall, however, a more general approach is needed, which would include clarity on the nature of incompatibility in cognition and how it arises.

A second related issue has been whether to adopt a dynamical vs. non-dynamical approach. This question arises in many areas of psychology (e.g., a decision researcher can adopt a non-dynamical heuristic vs. a drift diffusion model). However, in quantum cognition this choice is particularly significant, since it determines whether interference can arise from incompatibility (e.g., Busemeyer et al., 2011) or compatibility and 'mixing' amplitudes across two initially well-separate spaces (e.g., Pothos & Busemeyer, 2009; Trueblood & Busemeyer, 2011). This in turns affects whether there is an expectation of cooccurrence involving violations of the law of total probability and other fallacies, such as question order effects (cf. Boyer-Kassem et al., 2016; Yearsley & Trueblood, in press). Conceptually, there seems to be a clear distinction between the two situations, since interference due to incompatibility is about whether one question can chance the perspective or context for another one (Pothos et al., 2017), while interference due to mixing from dynamical evolution relates to whether two perspectives which individually recommend the same action are inconsistent with each other and so clash when brought together (as in failures of consequential reasoning in Shafir & Tversky, 1992). Operationally, distinguishing between these two situations awaits further work (cf. Gronchi & Strambini, 2017). Note, the adoption of dynamical models in cognitive psychology typically goes hand in hand with interest in modelling response times, but as yet there has been limited such application with quantum models (e.g., see Fuss & Navarro, 2013, who provided an open-systems quantum model for reaction time distribution in a two-alternative forced choice task).

Third, violations of Bell inequalities have led to controversy in psychology (as in physics). There are several difficulties. In physics, CHSH inequalities are derived under precise assumptions so that violations of these inequalities can be traced to the rejection of particular assumptions. This is essential so as to characterize the cause of Bell violations, particularly 'spooky action at a distance', with its far-reaching implications for space-time separability. In psychology, as we assume a classical brain, 'spooky action at a distance' is precluded. Dzhafarov et al.'s (2015, 2016; Dzhafarov & Kujala, 2013) generalization of the CHSH inequality enables a test of whether a CHSH/ Bell violation is due to signaling/ disturbing measurements vs. genuine contextuality. However, some investigators would be employing Bell as a test of compositionality and it may matter less whether the cause is signaling vs. contextuality (e.g., Bruza et al., 2015). Another difficulty is that a violation of Bell does not motivate a quantum model in the straightforward way that empirical evidence for an interference effect does. Even though quantum models can violate the Bell bound, this occurs only for certain states and for certain question pairs and so there is an additional onus for researchers wishing to make a link between Bell violation and a quantum model. Note, signaling is connectedness and it is easy to show violations of Bell in connected classical systems (Aerts, 1982).

Some other difficulties in creating quantum cognitive models are not unique to such models (Busemeyer et al., 2014; Yearsley & Busemeyer, 2016). A space of states needs to be specified; an initial state needs to be motivated, typically by assuming uniform uncertainty regarding the available options; a choice of static vs. dynamical approach needs be made; the relevant questions need to be represented. The latter can require more work for a quantum model than otherwise, since the precise relation between incompatible representations needs to be specified. Sometimes the approach is general and conditions for the required effect are examined across representational options (e.g., Busemeyer et al., 2011). Sometimes a specific representation is assumed, motivated from the form of the stimuli (e.g., White et al., 2014) or parameterized and fitted from empirical results (e.g., Trueblood et al., 2017). A recent advance has been the method of Busemeyer and Wang (in press) for extracting quantum representations from frequency tables violating classical constraints. In principle, this procedure can be adopted much like how e.g. categorization researchers have employed multidimensional scaling to extract vector representations (e.g., Nosofsky, 1984, 1992). A future challenge for this technique is how to accommodate representations beyond rays (Pothos et al., 2013).

In conclusion, there are many reasons to be optimistic about quantum cognitive models. Quantum theory provides a sophisticated technical framework for probabilistic inference, based on novel explanatory concepts, which complements modeling work using classical probability theory. To the extent that psychology researchers believe that part of cognition should be approached as probabilistic inference, it makes good sense to explore alternative probabilistic frameworks, and quantum theory is a possibility with considerable potential and promise.

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