The Co-Evolution of Network Structure, Strategic Behavior, and Stationary Equilibrium Dynamics Dynamics

Frank Page[1]
Department of Economics
Indiana University
Bloomington, IN 47405
fpage@indiana.edu

Joana Resende
Department of Economics and Cef.up
University of Porto
4200-464 Porto
Portugal
jresende@fep.up.pt

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[1]Also, Centre d'Economie de la Sorbonne, Universite Paris 1, Pantheon-Sorbonne.

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Abstract

We model the structure and strategy of social interactions prevailing at any point in time as a directed network and we address the following open question in the theory of social and economic network formation: given the rules of network and coalition formation, preferences of individuals over networks, strategic behavior of coalitions in forming networks, and the trembles of nature, what network and coalitional dynamics are likely to emergence and persist. Our main contributions are to formulate the problem of network and coalition formation as a dynamic, stochastic game and to show that: (i) the game possesses a stationary Markov equilibrium (in network and coalition formation strategies), (ii) together with the trembles of nature, this stationary equilibrium determines an equilibrium Markov process of network and coalition formation, and (iii) this endogenous Markov process possesses a finite set of ergodic measures, and generates a finite, disjoint collection of nonempty subsets of networks and coalitions, each constituting a basin of attraction. Moreover, we extend to the setting of endogenous Markov dynamics the notions of pairwise stability (Jackson-Wolinsky, 1996) and the path dominance core (Page-Wooders, 2009a). We show that in order for any network-coalition pair to emerge and persist, it is necessary that the pair reside in one of finitely many basins of attraction. The results we obtain here for endogenous network dynamics and stochastic basins of attraction are the dynamic analogs of earlier results on endogenous network formation and strategic basins of attraction in static, abstract games of network formation (Page and Wooders, 2009a), and build on the seminal contributions of Jackson and Watts (2002), Konishi and Ray (2003), and Dutta, Ghosal, and Ray (2005).

KEYWORDS: endogenous network dynamics, dynamic stochastic games of network formation, stationary Markov correlated equilibrium, equilibrium Markov process of network formation, basins of attraction, Harris decomposition, ergodic probability measures, dynamic path dominance core, dynamic pairwise stability.

JEL Classifications: A14, C71, C72

1 Introduction

In all social and economic interactions, individuals or coalitions choose not only with whom to interact but how to interact, and over time both the structure (the "with whom") and the strategy ("the how") of interactions change. Our objectives here are to model the structure *and* strategy of interactions prevailing at any point in time as a directed network and to shed new light on the co-evolution of network structure and strategic behavior by addressing the following open question in the theory of social and economic network formation: given rules of network formation, preferences of individuals over networks, strategic behavior of coalitions in forming networks, and trembles of nature, what *network and coalitional dynamics* are likely to emergence and persist. Thus, we propose to study the emergence of endogenous network and coalitional dynamics from strategic behavior and the randomness in nature.

Our main contributions are to formulate the problem of network formation as a dynamic, stochastic game, and to show that: (i) this game possesses an equilibrium in stationary Markov network and coalition formation strategies, (ii) together with the trembles of nature, these equilibrium strategies determine an equilibrium Markov process of network and coalition formation that respects the rules of network formation and the preferences of individuals and (iii) this equilibrium Markov process generates a finite, disjoint collection of nonempty subsets of networks and coalitions, each constituting a basin of attraction, and possesses a finite, nonempty set of ergodic measures.

In earlier work on the co-evolution of network structure and strategic behavior using static abstract games of network formation, Page and Wooders (2009) have shown that, given the rules of network formation and the preferences of individuals, these games possess *strategic basins of attraction* and these contain all networks that are likely to emerge and persist as the game unfolds. Moreover, they have shown that when any one of these strategic basins contains only one network, then that network (i.e., the single network contained in the singleton basin) is stable against all coalitional network deviation strategies - and thus the game has a nonempty *path dominance core*. Finally, Page-Wooders (2009) have shown that depending on how we specialize the rules of network formation and the dominance relation over networks, any network contained in the path dominance core is pairwise stable (Jackson-Wolinsky, 1996), strongly stable (Jackson-van den Nouweland, 2005), Nash (Bala-Goyal, 2000), or consistent (Chwe, 1994).

We show here that there are many parallels between the static abstract game formulation and the prior results of Page and Wooders for static games and the results we obtain here for our Markov dynamic game formulation. This is suggested already by the seminal paper by Jackson and Watts (2002) on the evolution of networks. Jackson and Watts present to our knowledge the first theory of stochastic dynamic network formation over a finite set of linking networks governed by a Markov chain generated by the myopic strategic behavior of players (following the Jackson-Wolinsky rules of network formation) and the trembles of nature. Their model builds on the earlier, nonstochastic model of dynamic network formation due to Watts (2001) - as far as we know, the first model of network dynamics (see also Skyrms and Pemantle, 2000)). By considering a sequence of perturbed irreducible and aperiodic Markov chains (i.e., each with a unique invariant measure) converging to the original Markov chain, they show that any pairwise stable network is necessarily contained in the support of an invariant measure - that is, in the support of a probability measure that places all its support on sets of networks likely to form in the long run. We show here that similar conclusions can be reached for directed networks with many arc types governed by arbitrary network formation rules.

In a general Markov game setting, with farsighted players, what precisely does it mean for a network to be pairwise stable - or stable in any sense? For example, if the state space of networks is large, then the endogenous Markov process of network formation is likely to have many invariant measures - and in fact many ergodic probability measures (i.e., measures that place all their probability mass on a single absorbing set). Which absorbing set contains networks stable in the sense of pairwise stability, or strong stability, or Nash stability? These are some of the questions we answer here in our study of endogenous network dynamics.

2 Primitives

2.1 The Space of Directed Networks

We begin by giving the formal definition of a directed network. Let N be a finite set of nodes with typical element denoted by i and let A be a finite set of arcs with typical element denoted by a. Arcs represent potential types of connections between nodes, and depending on the application, nodes can represent economic agents or economic objects such as markets or firms.

Definition 1 (*Directed Networks*)

Given node set N and arc set A, a directed network, G, is a nonempty subset of $A \times (N \times N)$.

The collection of all directed networks is denoted by $P(A \times (N \times N))$.

A directed network $G \in P(A \times (N \times N))$ thus consists of a set of ordered pairs of the form $(a, (i, i^{'}))$ where a is an arc type or an arc label and (i, i') is an ordered pair of nodes. We shall refer to any pair $(a, (i, i^{'})) \in G$ as a *connection* in network G. Thus, a network G is a finite set of connections specifying how the nodes in N are connected by the arcs in A. In a directed network order matters. In particular, $(a, (i, i^{'})) \in G$ means that nodes i and i' are connected by a type a arc *from* node i *to* node $i^{'}$. Because the set of nodes and the set of arcs (or arc types) are finite, the set of all possible directed networks $P(A \times (N \times N))$ is also finite.

Note that under our definition of a directed network, loops are allowed - that is, we allow an arc to go from a given node back to that given node.[1] Finally, note

[1] By allowing loops we are able to represent a network having no connections between distinct nodes as a network consisting entirely of loops at each node.

that under our definition an arc can be used multiple times in a given network and multiple arcs can go from one node to another. However, our definition does not allow an arc a to go from a node i to a node i' multiple times.

The following notation is useful in describing networks. Given directed network $G \in P(A \times (N \times N))$, let

$$G(a) := \left\{(i, i^{'}) \in N \times N : (a, (i, i^{'})) \in G\right\},$$

$$G(i, i') := \{a \in A : (a, (i, i')) \in G\}$$

$$G^{+}(i) := \left\{a \in A : (a, (i, i^{'})) \in G \text{ for some } i' \in N\ \right\},$$

and

$$G^{-}(i') := \left\{a \in A : (a, (i, i^{'})) \in G \text{ for some } i \in N\ \right\}.$$

Thus, in network G,

$G(a)$ is the *set of node pairs* connected by arc a,
$G(i, i')$ is the *set of arcs* from node i to node i',
$G^{+}(i)$ is the *set of arcs* leaving node i, and
$G^{-}(i')$ is the *set of arcs* entering node i'.

If for some arc $a \in A$, $G(a)$ is empty, then arc a is not used in network G. Also, if for some node $i \in N$, $G^{+}(i) \cup G^{-}(i)$ is empty, then node i is isolated.

In formulating our game of network and coalition formation, it will often be useful to restrict attention to a particular feasible subset of networks.

Definition 2 (*Feasible Networks*)

Given finite node set N and finite arc set A, a feasible set of networks is a nonempty, subset $\mathbb{G}$ of the collection of all directed networks $P(A \times (N \times N))$.

We will assume throughout that

A-1 (*finiteness of nodes and arcs*) the set of nodes N and arcs A are finite and that the feasible set of networks is given by a subset $\mathbb{G}$ of $P(A \times (N \times N))$.

Examples 1: Feasible Networks

(1) Club Networks:

Consider a collection of networks where some nodes represent players while other nodes represent clubs (or club locations) and where arc types represent the actions players take as members of clubs. In particular, let D be a finite set of players with typical element d, C be a finite set of club types (or club labels or club locations) with typical element c, and A be a finite set of arcs representing actions potentially available to all players with typical element a. For each player d and club c, let $A(d, c)$

be the feasible set of actions that can be taken by player d in club c. We adopt the convention that if $A(d,c)=\emptyset$, then player d cannot be a member of club c.

A club network G is a nonempty subset of $A\times(D\times C)$ such that (i) for all players $d\in D$, the section of G at d given by

$$G(d) := \{(a,c)\in A\times C : (a,(d,c))\in G\} \tag{1}$$

is nonempty; and (ii) for all $(a,(d,c))\in G$, $a\in A(d,c)$. Let $\mathbb{G}_K$ denote the collection of all such club networks. Note that by letting the set of nodes be given by $N=D\cup C$, we have

$$\mathbb{G}_K \subset P(A\times(N\times N)).$$

Also note that the condition $G(d)\neq\emptyset$ means that each player is a member of at least one club - and possibly more (i.e., membership in at least one club is required and multiple memberships are allowed).

An interesting special case of club networks is the collection of single membership networks. We denote this collection by $\mathbb{G}_{K1}$. A single membership club network G is a nonempty subset of $A\times(D\times C)$ such that (i) for all players $d\in D$, $G(d)$ contains one and only one element (i.e., $|G(d)|=1$, where $|G(d)|$ denotes the cardinality of $G(d)$), and (ii) for all $(a,(d,c))\in G$, $a\in A(d,c)$.

Specializing further, let the set of arc types be given by $A=\{1\}$, where $a=1$ denotes membership (i.e., $(1,(d,c))\in G$ means that player d is a member of club c in network G), and suppose there are three players

$$D=\{d_1,d_2,d_3\}$$

and two clubs

$$C=\{c_1,c_2\}.$$

Figure 1 depicts the single membership club network

$$G=\{(1,(d_1,c_1)),(1,(d_2,c_1)),(1,(d_3,c_2))\}.$$

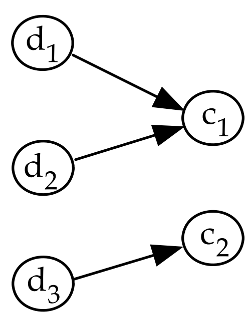

Figure 1: A Single Membership Club Network

(2) Marketing Networks:

Modifying example (2), suppose that the set of nodes N is given by $N:=F\cup M$, where F is a set of firms and M is a set of markets. Also, suppose that the set of arc types is given by a finite set of catalogs,

$$A:=\{C_1,\ldots,C_q\}.$$

Each catalog $C \in A$ is a nonempty, closed subset of $X \times P$, where $X \subset R^l$ is a closed bounded subset of product description vectors, P is a closed bounded interval of prices, and $(0,0) \in X \times P$.

Suppose that the feasible set of networks is given by

$$\mathbb{G}_M := \{G \subseteq A \times (F \times M) : \forall (f,m),\ |G(f,m)| \leq 1;\ \forall f,\ |G(f,m)| = 1 \text{ for some } m\}.$$

We call these networks, marketing networks. In marketing network $G \in \mathbb{G}_M$, a connection $(C,(f,m)) \in G$ means that firm f offers a catalog C of products and prices in market m. Note that feasibility requires that each firm offers a catalog (and only one catalog) in at least one market - but this catalog can be the "no contracting" catalog $C_0 = \{(0,0)\}$. In this way, firms are allowed to abstain from active participation.

Letting

$$F = \{f_1, f_2, f_3, f_4, f_5\}$$

be the set of firms and

$$M = \{m_1, m_2\}$$

be the set of markets, Figure 2 depicts a marketing network.

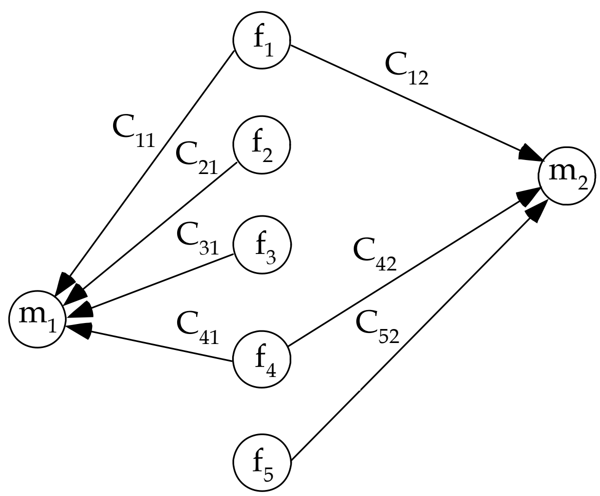

Figure 2: A Marketing Network

In this marketing network the connection $(C_{12},(f_1,m_2))$, with arc type $C_{12} \in A$, indicates that firm f_1 offers catalog C_{12} in market m_2.

2.2 Players and Coalitions

We will make a distinction between the set of players (or decision makers) and the set of nodes. In particular, we will not assume that the set of players and the set of nodes are necessarily one and the same. As in the marketing network example (2) above, some nodes are firms (i.e., players or decision makers) while other nodes are markets (i.e., are not players and are passive).

Because changing one network to another network very often involves groups of players acting in concert, coalitions will play an important role in our model. Let D denote the set of players (a set not necessarily equal to N the set of nodes) with typical element denoted by d and let $P(D)$ denote the collection of all player coalitions (i.e., nonempty subsets of D) with typical element denoted by S. We will assume that the set of players D has cardinality m (i.e., $|D| = m$). Depending on the rules of network formation, it will often be useful to restrict attention to a particular feasible subset of coalitions.

Definition 3 (*Feasible Coalitions*)

Given player set D, a feasible set of coalitions is a nonempty subset $\mathcal{F}$ of the collection of all coalitions $P(D)$.

We will assume throughout that

A-2 (*finiteness of players*) the set of players D is finite and that the feasible set of coalitions is given by a subset $\mathcal{F}$ of $P(D)$.

Examples 2: Feasible Coalitions

(1) Suppose that the feasible set of coalitions is given by

$$\mathcal{F}_2 = \{S \in P(D) : |S| \leq 2\}.$$

Thus, all feasible coalitions consist of at most two players. The set $\mathcal{F}_2$ is, for example, the feasible set for the Jackson-Wolinsky rules of network formation (Jackson-Wolinsky (1996)). If the set of nodes and the set of players are one in the same, then under the Jackson-Wolinsky rules, a connection can be removed from a network if and only if one or both players involved in the connection agree to remove the connection (arc subtraction can be unilateral), and a connection can be added to a network if and only if both players involved in the connection agree to add the connection (arc addition is bilateral).

(2) Suppose that the feasible set of coalitions is given

$$\mathcal{F}_1 = \{S \in P(D) : |S| = 1\}.$$

Thus, all feasible coalitions consist of one player. The set $\mathcal{F}_1$ is, for example, the feasible set for the noncooperative, Bala-Goyal rules (Bala-Goyal (2000)). If the set of nodes and the set of players are one in the same, then under the Bala-Goyal rules, a connection can be added or subtracted from a network if and only if the initiating player in the connection agrees to add or subtract the connection (arc addition and subtraction is unilateral).[2]

(3) Consider the following club network example which illustrates another aspect of the usefulness of making a distinction between nodes and players. First, suppose

[2] Let $(a, (i, i'))$ be a connection in network G where the set of nodes is equal to the set of players. In the connection $(a, (i, i'))$, player i is the initiating player.

that the set of nodes is given by $N = I \cup C$, where $I = \{i_1, i_2, \ldots, i_L\}$ is a set of individuals and $C = \{c_1, c_2, \ldots, c_K\}$ is a set of clubs, and assume that the set of arcs is given by $A = \{a_1, a_2, \ldots, a_M\}$. A typical connection in a club network is given by

$$(a, (i, c)),$$

where $a \in A$, $i \in I$, $c \in C$, and $(a, (i, c))$ means that individual i is a member of club c and takes action a in club c.

Next, suppose that the set of players is given by $D = P(I)$. Thus, a player $d \in D$ is a group or coalition of individuals. Finally, assume that the feasible set of *player* coalitions is given by

$$\mathcal{F}_2 = \{S \in P(D) : |S| \leq 2\}.$$

Thus, each player coalition consists of at most 2 players and each player is a group of individuals.

2.3 States, Actions, and Payoffs

2.3.1 States

We shall take as the state space the set $\Omega := (\mathbb{G} \times \mathcal{F})$ of all feasible network-player coalition pairs. Each state in $(\mathbb{G} \times \mathcal{F})$ has the following interpretation: if (G, S) is the current state, then G is the current status quo network of social interactions and it is player coalition S's turn to propose a new status quo network.

In order to save writing and spare the reader, when no confusion is possible, we will use Ω to denote the state space $\mathbb{G} \times \mathcal{F}$ and ω to denote an elements (G, S) of the state space. Thus, we will use the notation

$$\Omega := \mathbb{G} \times \mathcal{F}$$
and
$$\omega := (G, S).$$

2.3.2 Actions

In our game each player's action takes the form of a network recommendation or network proposal. In particular, given current state $\omega \in \Omega$, each player $d \in D$ has available a nonempty subset of network proposals $\Phi_d(\omega) \subseteq \mathbb{G}$ that can be put forth by player d for consideration by nature. However, only players who are members of the status quo coalition (i.e., the coalition whose turn it is to move) are allowed to propose *substantive* changes and each such proposal must be consistent with the rules of network formation. In particular, if $G' \in \Phi_d(G, S)$ is proposed by player $d \in S$ (and therefore, by a member of the status quo player coalition), then the proposed network G' must be such that under the rules of network formation it is possible for coalition S or some subcoalition $S' \subseteq S$, to which player d belongs to change the status quo network G to network G'. Moreover, because players who are not members of the status quo coalition are not allowed to propose substantive changes, these players (i.e., players $d \notin S$) can only propose that the status quo network be maintained. Formally, we will assume that

A-3 (*properties of constraint mappings*)

for each player $d \in D$, the correspondence $\Phi_d(\cdot)$ is such that, for all $\omega = (G, S)$

$$\left.\begin{array}{c}\text{(a) } G \in \Phi_d(G,S) \text{ for all } d \in D, \\ \text{and} \\ \text{(b) } \{G\} = \Phi_d(G,S) \text{ for all } d \notin S.\end{array}\right\} \tag{2}$$

Thus, under A-1(a) each player d in each state has the option of proposing that the status quo network be maintained and under A-1(b) if the player is not part of the status quo coalition, then the status quo is the *only* network proposal available to that player. Moreover, if network $G' \in \Phi_d(G,S)$ is proposed by player $d \in S$, then under the rules of network formation, it must be feasible for player d, working alone or together with some subcoalition $S' \subseteq S$ (including possibly all members of S), to change the status quo network G to the proposed network G'.

We will denote by $\Phi(\cdot)$ the aggregate constraint correspondence,

$$\omega \to \Phi(\omega) := \Pi_{d \in D} \Phi_d(\omega). \tag{3}$$

2.3.3 Payoffs

In order for players to decide which networks to propose, we must specify player payoff functions. We shall assume that

A-4 (*payoff functions*)

each player $d \in D$ has a payoff function

$$r_d(\cdot, \cdot) : \Omega \times \mathbb{G}^m \to [-M, M]. \tag{4}$$

Thus, if the current state is $\omega = (G, S)$ (i.e., if the status quo network is G and it is coalition S's turn to move) and if players propose m-tuple of networks $G_D := (G_d)_{d \in D} \in \Phi(\omega)$, then player $\overline{d}'s$ payoff is given by

$$r_{\overline{d}}(\omega, G_D) := r_{\overline{d}}(\omega, (G_{\overline{d}}, G_{-\overline{d}})).$$

2.4 The Law of Motion and the Proposal-Dependent Markov Transition Matrix

In order to further simplify the notation, let

$$\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\}, \tag{5}$$

and let

$$H := \{1, 2, \ldots, N\} \tag{6}$$

with typical elements i and j index states. Thus, $i \in H$ if and only if $\omega_i \in \Omega$, and for any nonempty subset E of Ω, $i \in H_E$ if and only if $\omega_i \in E$. Often we will use i and j to denote states, rather than ω $(= \omega_i)$ and ω' $(= \omega_j)$.

Given the profile of player proposals $G_D = (G_d)_{d \in D} := (G_d, G_{-d})$ and given the current state, $\omega = (G, S)$, *nature* then *chooses* the next state (i.e., the next network-coalition pair) in accordance with the *Markov transition law*, $q(\cdot|\cdot, G_D)$. Thus, given m-tuple of player proposals G_D and current state ω, the probability with which nature chooses the next state ω' is given by $q(\omega'|\omega, G_D)$. For each proposal m-tuple G_D, let $Q(G_D)$ be the resulting $N \times N$ Markov transition matrix. The transition matrix $Q(G_D)$ has typical entry

$$q_{ij}(G_D) := q(\omega_j|\omega_i, G_D) \tag{7}$$

where $q_{ij}(G_D)$ is the probability that nature moves from state $\omega_i = (G_i, S_i)$ to state $\omega_j = (G_j, S_j)$ given player proposals G_D.

2.5 Plans and Stationary Strategies

A *plan* $\pi_d = (\pi_d^0, \pi_d^1, \ldots)$ for player $d \in D$ is a sequence of history dependent conditional probability measures on the feasible set of networks $\mathbb{G}$. Under plan π_d at time point n given the history of states and proposal m-tuples, $H^{n-1} := \left(\omega^0, G_D^0, \omega^1, G_D^1, \ldots, \omega^{n-1}, G_D^{n-1}\right)$, and given the current (time point n) state $\omega^n = (G^n, S^n)$, player d chooses a network proposal according to the conditional probability measure

$$\pi_d^n(\cdot|H^{n-1}, \omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right). \tag{8}$$

Here, $\mathcal{P}\left(\Phi_d(\omega^n)\right)$ is the set of all probability measures with support contained in $\Phi_d(\omega^n)$. Let $\mathcal{H}^{n-1}$ denote set of all (n)-histories and let

$$\Pi_d^n := \Pi_{\Phi_d}(\mathcal{H}^{n-1} \times \Omega, \mathcal{P}(\Omega))$$

denote the set of all measurable functions, $(H^{n-1}, \omega^n) \to \pi_d^n(\cdot|H^{n-1}, \omega^n) \in \mathcal{P}(\Omega)$ such that $\pi_d^n(\cdot|H^{n-1}, \omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right)$ for all $\omega^n \in \Omega$. Formally, the set of plans for player d is given by

$$\Pi_d^\infty := \prod_{n=1}^{\infty} \Pi_d^n.$$

A *Markov plan* $\psi_d = (\psi_d^1, \psi_d^2, \ldots)$ for player $d \in D$ is a sequence of state-dependent conditional probability measures on Ω. Under Markov plan ψ_d, at time point n, given the current (time point n) status quo network-coalition pair (or state) $\omega^n = (G^n, S^n)$, player d chooses a network proposal according to the conditional probability measure

$$\psi_d^n(\cdot|\omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right). \tag{9}$$

Let

$$\Sigma_d^n := \Sigma_{\Phi_d}(\Omega, \mathcal{P}(\Omega)) := \Sigma_{\Phi_d}$$

denote the set of all measurable functions, $\omega \to \psi_d^n(\cdot|\omega) \in \mathcal{P}(\Omega)$ such that $\psi_d^n(\cdot|\omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right)$ for all $\omega^n \in \Omega$. The set of Markov plans for player d is given by

$$\Sigma_d^\infty := \prod_{n=1}^{\infty} \Sigma_d^n.$$

A *stationary Markov plan* $(\sigma_d, \sigma_d, \ldots)$ for player $d \in D$ - or as we shall call it here, a stationary strategy for player $d \in D$ - is a constant sequence of state-dependent conditional probability measures on Ω. Under stationary strategy $(\sigma_d, \sigma_d, \ldots)$ given the current (time point n) status quo network-coalition pair (or state) $\omega^n = (G^n, S^n)$, player d, at each and every time point n, chooses a network proposal according to the conditional probability measure

$$\sigma_d(\cdot|\omega^n) \in \mathcal{P}\left(\Phi_d(\omega^n)\right). \tag{10}$$

Rather than write $\sigma_d(\cdot|\omega)$ we will sometimes write $\sigma_d(\omega)$.

A *pure stationary strategy* for player $d \in D$ is a stationary Markov strategy $(\sigma_d, \sigma_d, \ldots)$ such that for some function

$$\begin{aligned} f_d(\cdot) : \Omega \to \mathbb{G} \text{ with } f_d(\omega) \in \Phi_d(\omega) \text{ for all } \omega \in \Omega, \\ \sigma_d(f_d(\omega)|\omega) = 1 \text{ for all } \omega \in \Omega. \end{aligned} \tag{11}$$

Thus under *pure stationary strategy* $(\sigma_d, \sigma_d, \ldots)$ in any state $\omega \in \Omega$, the conditional probability measure for player d assigns probability 1 to the network proposal $f_d(\omega) \in \Phi_d(\omega)$.[3] Rather than represent a pure stationary strategy for player $d \in D$ using a conditional probability measure $\sigma_d(\cdot|\omega)$ concentrating all its probability mass on a particular state dependent network $f_d(\omega)$, we will often times instead represent a pure stationary strategy for player $d \in D$ using the underlying function $f_d(\cdot)$. Thus, a pure stationary strategy for player $d \in D$ will often times be described as a constant sequence of functions $(f_d, f_d, \ldots)$ such that for all $\omega \in \Omega . f_d(\omega) \in \Phi_d(\omega)$ and $\sigma_d(f_d(\omega)|\omega) = 1$.

2.6 Player Payoffs

Given m-tuple of stationary strategies $(\sigma_d(\cdot|\cdot))_{d \in D}$, if the current state is $\omega \in \Omega$ then player d's immediate expected payoff is

$$r_d(\omega, \sigma_D(\omega)) := \sum_{G_D \in \Phi(\omega)} r_d(\omega, G_D)\sigma_D(G_D|\omega)$$

where $\sigma_D(\omega) := \sigma_D(\cdot|\omega)$ is the product measure $\times_d \sigma_d(\cdot|\omega)$ with support contained in $\Phi(\omega) := \Pi_{d \in D}\Phi_d(\omega) \subseteq \mathbb{G}^m$.

If network proposal m-tuple G_D is chosen according to product measure $\sigma_D(\cdot|\omega)$, then nature chooses the next network-coalition pair (i.e., the next state) according to the probability measure $q(\cdot|\omega, G_D)$.

Let

$$r_d^n(\sigma_D)(\omega)$$

$$:= \begin{cases} \sum_{G_D \in \Phi(\omega)} r_d(\omega, G_D)\sigma_D(G_D|\omega) & \text{for } n = 0 \\ \sum_{\omega' \in \Omega} \left[\sum_{G_D \in \Phi(\omega)} r_d(\omega', G_D)\sigma_D(G_D|\omega')\right] q^{(n)}(\omega'|\omega, \sigma_D(\omega)) & \text{for } n \geq 1, \end{cases}$$

[3] Such a conditional probability measure is often denoted using a Dirac measure notation, $\delta_{f_d(\omega)}$.

denote the n^{th} period expected payoff to player d under stationary strategy $\sigma_D(\cdot)$ starting at network-coalition pair $\omega = (G, S)$ given law of motion $q(\cdot|\cdot, \sigma_D(\cdot))$.[4] Here, for $n \geq 1$, $q^n(\cdot|\omega, \sigma_D(\omega))$ is defined recursively by

$$q^{(n)}(E|\omega, \sigma_D(\omega))$$
$$= \sum_{\omega' \in \Omega} q^{(n-1)}(E|\omega', \sigma_D(\omega'))q(\omega'|\omega, \sigma_D(\omega))$$
$$= \sum_{\omega' \in \Omega} q^{(n-1)}(\omega'|\omega, \sigma_D(\omega))q(E|\omega', \sigma_D(\omega')).$$

The discounted expected payoff to player d over an infinite time horizon under stationary strategy $\sigma_D(\cdot) \in \prod_{d\in D} \Sigma_{\Phi_d}$ starting at state ω is then given by

$$E_d(\sigma_D)(\omega) := \sum_{n=0}^{\infty} \beta_d^n r_d^n(\sigma_D)(\omega).$$

In general, the discounted expected payoff to player d over an infinite time horizon under plan $\pi_D = (\pi_d)_{d\in D} \in \Pi^\infty := \prod_{d\in D} \Pi_d^\infty$ starting in state ω is then given by

$$E_d(\pi_D)(\omega) := \sum_{n=0}^{\infty} \beta_d^n r_d^n(\pi_D)(\omega).$$

3 Dynamic Network Formation Games

3.1 Existence of Nash Equilibrium in Stationary Strategies

A dynamic network formation game is given by

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^\infty)_{d\in D}.$$

A dynamic network formation game starting at state $\omega \in \Omega$ is given by

$$\Gamma_\omega := (\Omega, E_d(\cdot)(\omega), \Pi_d^\infty)_{d\in D}.$$

[4] We will regularly abuse our own notation by using

$$\sigma_D(\cdot) := \sigma_D(\cdot|\cdot)$$

to denote the conditional product probability measure

$$\times_d \sigma_d(\cdot) := \times_d \sigma_d(\cdot|\cdot)$$

as well as to denote the m-tuple of conditional probability measures

$$(\sigma_d(\cdot), \sigma_{-d}(\cdot)) := (\sigma_d(\cdot|\cdot), \sigma_{-d}(\cdot|\cdot)).$$

Thus, depending on the context,

$$\sigma_D(\cdot) := \times_d \sigma_d(\cdot|\cdot)$$
or
$$\sigma_D(\cdot) := (\sigma_d(\cdot), \sigma_{-d}(\cdot)),$$

and of course, we claim that the meaning will be clear from the context.

Definition 4 (*Nash Equilibrium*)

A stationary strategy $(\sigma_d^*(\cdot|\cdot))_{d\in D}$ *with corresponding m-tuple of stationary strategies* $\sigma_D^*(\cdot) = (\sigma_d^*(\cdot|\cdot))_{d\in D}$ *is a Nash equilibrium of the dynamic network formation game* Γ *if for all starting network-coalition pairs* $\omega = (G, S) \in \mathbb{G}\times\mathcal{F}$ *and all players* $d \in D$,

$$E_d(\sigma_d^*, \sigma_{-d}^*)(\omega) \geq E_d(\pi_d, \sigma_{-d}^*)(\omega) \text{ for all } \pi_d \in \Pi_d^\infty.$$

Thus, an m-tuple of stationary strategies $(\sigma_d^*(\cdot|\cdot))_{d\in D}$ is a Nash equilibrium of dynamic network formation game Γ if it is a Nash equilibrium for the game Γ_ω for all starting states.

Theorem 1 (*The Existence of Nash Equilibrium in Stationary Strategies*)

Under assumptions [A-1]-[A-4] the dynamic network formation game,

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^\infty)_{d\in D},$$

has a Nash equilibrium in stationary strategies.

Theorem 1 is an immediate consequence of Theorem 1 in Federgruen (1978). Moreover, letting

$$w_d^*(\omega) := E_d(\sigma_d^*, \sigma_{-d}^*)(\omega),$$

by Theorem 6f in Blackwell (1968), $(\sigma_d^*(\cdot|\cdot))_{d\in D}$ is a Nash equilibrium in stationary strategies if and only if

$$w_d^*(\omega) = \max_{\sigma'\in\mathcal{P}(\Phi_d(\omega))}\left(r_d(\omega, (\sigma', \sigma_{-d}^*(\omega)) + \beta_d \sum_{\omega'\in\Omega} w_d^*(\omega')q(\omega'|\omega, (\sigma', \sigma_{-d}^*(\omega))\right),$$

for all $\omega \in \Omega$ and $d \in D$. The quantity, $w_d^*(\omega)$, is the present value to player d of following his stationary strategy, $\sigma_d^*(\cdot)$, in proposing networks in all future periods, starting in state ω and assuming that all other players also follow their stationary strategies, $\sigma_{-d}^*(\cdot)$, in proposing networks in all future periods. We will refer to the function $w_d^*(\cdot) : \Omega \to [-M, M]$ as player $d's$ value function and we will write $w_D(\cdot)$ to denote the m-tuple of value functions, $(w_d(\cdot))_{d\in D}$ - and we will write $w_d^*(\cdot)$ to denote player $d's$ equilibrium value function.

3.2 Characterization of Stationary Equilibria

In addition to the characterization of stationary equilibria due to Blackwell (1968), there is also a nonlinear programming characterization of stationary equilibria due to Filar, Schultz, Thuijsman, and Vrieze (1991).[5] We state their characterization

[5] Herings and Peeters (2004) also provide a nonlinear programming characterization of stationary equilibria (see Theorem 3.6, page 40, in Herings and Peeters 2004).

result here for the convenience of the reader, and in Example 3, we will use their characterization result to construct an example of a pure stationary equilibrium in a dynamic club network formation game.

To begin, let β_d be player $d's$ discount rate and let

$$(w_D(\cdot), \sigma_D(\cdot)) := (w_d(\cdot), \sigma_d(\cdot|\cdot))_{d \in D}$$

denote an m-tuple of value function-stationary strategy pairs where for each player d

$$w_d(\cdot) : \Omega \to [-M, M] \text{ and } \sigma_d(\cdot) : \Omega \to \mathcal{P}(\Omega),$$

such that $\sigma_d(\omega) \in \mathcal{P}(\Phi_d(\omega))$ for all $\omega \in \Omega$.

Theorem 2 (*A Nonlinear Programming Characterization of Stationary Equilibria*)

Suppose that assumptions [A-1]-[A-4] hold. A valuation function, stationary strategy profile pair $(w_D^*(\cdot), \sigma_D^*(\cdot))$ solves the dynamic network formation game Γ if and only if $(w_D^*(\cdot), \sigma_D^*(\cdot))$ solves

$$\min \sum_{d \in D} \sum_{i \in H} \left[w_d(\omega_i) - r_d(\omega_i, \sigma_D(\omega_i)) - \beta_d \sum_{j \in H} w_d(\omega_j) q(\omega_j | \omega_i, \sigma_D(\omega_i)) \right] \quad (12)$$

over $\sigma_D(\cdot)$ such that for all $d \in D$ and $\omega_i \in \Omega$, $\sigma_d(\omega_i) \in \mathcal{P}(\Phi_d(\omega_i))$ and over $w_D(\cdot)$ such that for all $d \in D$, $\omega_i \in \Omega$, and $G_d \in \Phi_d(\omega_i)$

$$w_d(\omega_i) \geq r_d(\omega_i, G_d, \sigma_{-d}(\omega_i)) + \beta \sum_{j \in H} w_d(\omega_j) q(\omega_j | \omega_i, G_d, \sigma_{-d}(\omega_i))$$

Moreover, if $(w_D^*(\cdot), \sigma_D^*(\cdot))$ solves (12), then letting

$$m^* := \sum_{d \in D} \sum_{i \in H} \left[w_d^*(\omega_i) - r_d(\omega_i, \sigma_D^*(\omega_i)) - \beta \sum_{j \in H} w_d^*(\omega_j) q(\omega_j | \omega_i, \sigma_D^*(\omega_i)) \right],$$

$m^* \geq 0$, and if $m^* > 0$, then $\sigma_D^*(\cdot)$ is an ε-equilibrium with $\varepsilon \in (0, \frac{m^*}{1-\beta})$.

Example 3: Pure Stationary Equilibrium in a Dynamic Club Network Formation Game

First, as in Example 2(3), assume that the set of nodes is given by $N = I \cup C$, where $I = \{w_1, w_2, b\}$ is a set of individuals and $C = \{c_1, c_2\}$ is a set of clubs and suppose that the set of arcs is given by $A = \{1\}$, where $a = 1$ denotes club membership by an individual. Thus, a typical connection is given by $(1, (i, c))$, where $i \in I$, $c \in C$, and $(1, (i, c))$ means that individual i is a member of club c.

Next, suppose that the set of players is given by

$$D = \{d_1, d_2\} = \{\{w_1, b\}, \{w_2, b\}\}.$$

Thus, a player d_i is a group of individuals and here player d_1 is the group $\{w_1, b\}$, while player d_2 is the group $\{w_2, b\}$.

Finally, assume that the feasible set of *player* coalitions is given by

$$\mathcal{F}_1 = \{\{d_1\}, \{d_2\}\}.$$

Thus, each player coalition consists of a single player and each player is a group of individuals. Note that individual w_1 is in group d_1 but not in group d_2 and that individual w_2 is in group d_2 but not in group d_1 (i.e., $w_i \in d_i$ for all i), while individual b is in both groups. We will assume that whenever it is coalition $\{d_i\}'s$ turn to move (i.e., whenever $\{d_i\}$ is the status quo coalition), then player d_i (i.e., group $\{w_i, b\}$) is represented by individual w_i (the group leader). In particular, we will assume that individual w_i has complete control over the membership moves of all members of group d_i (i.e., w_i has complete control over his membership moves and the membership moves of individual b). But these moves are subject to the following constraints:

(1) individual w_1 cannot be a member of club c_2.

(2) individual b can be a member of club c_1 if and only if individuals w_1 and w_2 are members of club c_1.

The set of all single individual membership club networks $\mathbb{G}_{K1}$ is given by the collection of all nonempty subsets G of $A \times (I \times C)$ such that for all $i \in I$, $|G(i)| = 1$. The set of all such club networks satisfying conditions (1) and (2) is the subset of $\mathbb{G}_{K1}$ given by

$$\mathbb{G} = \{G_1, G_2, G_3\},$$

with networks as depicted in Figure 3.

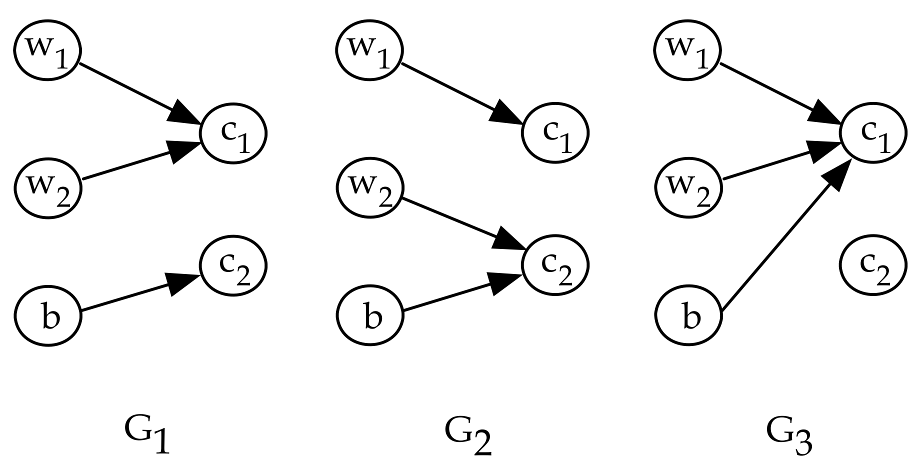

Figure 3: The Feasible Set of Club Networks

The set of states is given by

$$\Omega = \left\{ \underbrace{(G_1,\{d_1\})}_{\omega_1}, \underbrace{(G_1,\{d_2\})}_{\omega_2}, \underbrace{(G_2,\{d_1\})}_{\omega_3}, \underbrace{(G_2,\{d_2\})}_{\omega_4}, \underbrace{(G_3,\{d_1\})}_{\omega_5}, \underbrace{(G_3,\{d_2\})}_{\omega_6} \right\}.$$

Table 1 below lists players' state-contingent network proposal constraint sets.

	$\Phi_{d_1}(\cdot)$	$\Phi_{d_2}(\cdot)$
ω_1	$\{G_1, G_3\}$	$\{G_1\}$
ω_2	$\{G_1\}$	$\{G_1, G_2, G_3\}$
ω_3	$\{G_2\}$	$\{G_2\}$
ω_4	$\{G_2\}$	$\{G_1, G_2, G_3\}$
ω_5	$\{G_1, G_3\}$	$\{G_3\}$
ω_6	$\{G_3\}$	$\{G_1, G_2, G_3\}$

Table 1: Player's State-Contingent Constraint Sets

For example, in state $\omega_5 = (G_3, \{d_1\})$ player d_1 has available network proposals G_1 and G_3 (i.e., $\Phi_{d_1}(\omega_5) = \{G_1, G_3\}$).

We will assume that each player's payoff function,

$$r_d(\cdot,\cdot) : \Omega\times\mathbb{G}^m \to [-M, M],$$

depends only on the status quo state. Thus, for all players $d \in D$ and for all state-network proposal pairs,

$$(\omega, G_D) = (\underbrace{(G,\{d'\})}_{\omega}, G_D) \in \Omega\times\mathbb{G}^m,$$

payoffs are given by

$$r_d(\omega, G_D) = v_d(\omega),$$

where $v_d(\cdot)$ is the total payoff to player d in state $\omega = (G, \{d'\})$. Table 2 lists each player's state-contingent payoffs.

	$\underbrace{(G_1,\{d_1\})}_{\omega_1}$	$\underbrace{(G_1,\{d_2\})}_{\omega_2}$
d_1	$v_{d_1}(\omega_1) = 3$	$v_{d_1}(\omega_2) = 3$
d_2	$v_{d_2}(\omega_1) = 2$	$v_{d_2}(\omega_2) = 2$

	$\underbrace{(G_2,\{d_1\})}_{\omega_3}$	$\underbrace{(G_2,\{d_2\})}_{\omega_4}$
d_1	$v_{d_1}(\omega_3) = 2$	$v_{d_1}(\omega_4) = 2$
d_2	$v_{d_2}(\omega_3) = 1$	$v_{d_2}(\omega_4) = 1$

	$\underbrace{(G_3,\{d_1\})}_{\omega_5}$	$\underbrace{(G_3,\{d_2\})}_{\omega 6}$
d_1	$v_{d_1}(\omega_5) = 1$	$v_{d_1}(\omega_6) = 1$
d_2	$v_{d_2}(\omega_5) = 3$	$v_{d_2}(\omega_6) = 3$

Table 2: State-Contingent Player Payoffs

Note that state-contingent payoffs are in fact invariant with respect to the coalition, $\{d_1\}$ or $\{d_2\}$, whose turn it is to move. For example, according to Table 2, in both states ω_1 and ω_2 player d_2 receives payoff $v_{d_2}(\omega_1) = 2$. In vector form the state-contingent payoffs to each player are given by

$$\left.\begin{array}{c} r_{d_1} = (r_{d_1}(\omega_1, G_D), \ldots, r_{d_1}(\omega_6, G_D)) \\ = (v_{d_1}(\omega_1), \ldots, v_{d_1}(\omega_6)) \\ = (3, 3, 2, 2, 1, 1) \\ \text{and} \\ r_{d_2} = (r_{d_2}(\omega_1, G_D), \ldots, r_{d_2}(\omega_6, G_D)) \\ = (v_{d_2}(\omega_1), \ldots, v_{d_2}(\omega_6)) \\ = (2, 2, 1, 1, 3, 3). \end{array}\right\} \quad (13)$$

Using the nonlinear programming characterization of stationary equilibria given in Theorem 2, we will show that the *pure* stationary strategies for players d_1 and d_2 given in Table 3 are Nash equilibrium proposal strategies for the dynamic game of club network formation.

states		$f^*_{d_1}(\cdot)$	$f^*_{d_2}(\cdot)$
ω_1	$= (G_1, \{d_1\})$	$f^*_{d_1}(\omega_1) = G_1$	$f^*_{d_2}(\omega_1) = G_1$
ω_2	$= (G_1, \{d_2\})$	$f^*_{d_1}(\omega_2) = G_1$	$f^*_{d_2}(\omega_2) = G_3$
ω_3	$= (G_2, \{d_1\})$	$f^*_{d_1}(\omega_3) = G_2$	$f^*_{d_2}(\omega_3) = G_2$
ω_4	$= (G_2, \{d_2\})$	$f^*_{d_1}(\omega_4) = G_2$	$f^*_{d_2}(\omega_4) = G_3$
ω_5	$= (G_3, \{d_1\})$	$f^*_{d_1}(\omega_5) = G_1$	$f^*_{d_2}(\omega_5) = G_3$
ω_6	$= (G_3, \{d_2\})$	$f^*_{d_1}(\omega_6) = G_3$	$f^*_{d_2}(\omega_6) = G_3$

Table 3: Pure Strategies

For example, according to Table 3, in state ω_4, player d_2 proposes feasible network $f^*_{d_2}(\omega_4) = G_3$. Using the state-contingent proposal constraint sets in Table 1, we can easily list the *feasible deviations* from the equilibrium proposals given in Table 3.

states		
ω_1	$f^*_{d_1}(\omega_1) = G_1 \to G_3$	$f^*_{d_2}(\omega_1) = G_1$
ω_2	$f^*_{d_1}(\omega_2) = G_1$	$f^*_{d_2}(\omega_2) = G_3 \to G_1, G_2$
ω_3	$f^*_{d_1}(\omega_3) = G_2$	$f^*_{d_2}(\omega_3) = G_2$
ω_4	$f^*_{d_1}(\omega_4) = G_2$	$f^*_{d_2}(\omega_4) = G_3 \to G_1, G_2$
ω_5	$f^*_{d_1}(\omega_5) = G_1 \to G_3$	$f^*_{d_2}(\omega_5) = G_3$
ω_6	$f^*_{d_1}(\omega_6) = G_3$	$f^*_{d_2}(\omega_6) = G_3 \to G_1, G_2$

Table 4: Feasible Deviations

For example, according to Table 4, in state $\omega_2 = (G_1, \{d_2\})$ player 1 proposes $f^*_{d_1}(\omega_2) = G_1$ and has no feasible deviations (because it is not player 1's turn to move, player 1 can only propose the status quo G_1). However, in state $\omega_2 = (G_1, \{d_2\})$, player 2 proposes $f^*_{d_2}(\omega_2) = G_3$ and has two feasible deviations, G_1 and G_2.

We will assume that the law of motion, $q(\cdot|\cdot,\cdot)$ is such that given any status quo state, only states containing networks proposed by one of the players are assigned positive probabilities. For example, if players d_1 and d_2 both propose network G_2, then starting at *any* status quo state, say $\omega = (G, \{d\})$, only states

$$\underbrace{(G_2, \{d_1\})}_{\omega_3} \text{ and } \underbrace{(G_2, \{d_2\})}_{\omega_4},$$

are assigned positive probability. Let $\Omega(G_{d_j})$ denote the set of states containing the network G_{d_j} proposed by player d_j. Thus, under the law of motion, only states contained in $\Omega(G_{d_1}) \cup \Omega(G_{d_2})$ are assigned positive probabilities.

Further, we will assume that these probabilities are given by

$$q(\omega' | \omega, (G_{d_j})_{d_j}) = \frac{\sum_{d_j \in D} \left[e^{\left(v_{d_j}(\omega') - v_{d_j}(\omega)\right)} I_{\Omega(G_{d_j})}(\omega') \right]}{\sum_{\omega'' \in \Omega} \sum_{d_j \in D} \left[e^{\left(v_{d_j}(\omega'') - v_{d_j}(\omega)\right)} I_{\Omega(G_{d_j})}(\omega'') \right]}, \tag{14}$$

where for states $\omega \in \Omega$

$$I_{\Omega(G_{d_j})}(\omega) = \begin{cases} 1 & \text{if } \omega \in \Omega(G_{d_j}) \\ 0 & \text{if } \omega \notin \Omega(G_{d_j}). \end{cases}$$

For example, if the status quo state is $\omega_3 = (G_2, \{d_1\})$ and both players propose network G_2, then the probability that state ω_3 occurs (i.e., that the process stays in state ω_3) is given by

$$q(\omega_3 | \omega_3, G_2, G_2)$$

$$= \frac{\left[e^{\left(v_{d_1}(\omega_3) - v_{d_1}(\omega_3)\right)} \right] + \left[e^{\left(v_{d_2}(\omega_3) - v_{d_2}(\omega_3)\right)} \right]}{\left[e^{\left(v_{d_1}(\omega_3) - v_{d_1}(\omega_3)\right)} \right] + \left[e^{\left(v_{d_1}(\omega_4) - v_{d_1}(\omega_3)\right)} \right] + \left[e^{\left(v_{d_2}(\omega_3) - v_{d_2}(\omega_3)\right)} \right] + \left[e^{\left(v_{d_2}(\omega_4) - v_{d_2}(\omega_3)\right)} \right]}$$

$$= \frac{[e^0]}{[e^0] + [e^0] + [e^0] + [e^0]} = .5.$$

This probability is higher than it would have been had player 2, for example, proposed instead network G_1 - because G_1 is not the network in state $\omega_3 = (G_2, \{d_1\})$. Specifically, we would have

$$q(\omega_3 | \omega_3, G_2, G_1)$$

$$= \frac{\left[e^{\left(v_{d_1}(\omega_3) - v_{d_1}(\omega_3)\right)} I_{\Omega(G_2)}(\omega_3) \right]}{\left[e^{\left(v_{d_1}(\omega_3) - v_{d_1}(\omega_3)\right)} \right] + \left[e^{\left(v_{d_1}(\omega_4) - v_{d_1}(\omega_3)\right)} \right] + \left[e^{\left(v_{d_2}(\omega_1) - v_{d_2}(\omega_3)\right)} \right] + \left[e^{\left(v_{d_2}(\omega_2) - v_{d_2}(\omega_3)\right)} \right]}$$

$$= \frac{[e^0]}{[e^0] + [e^0] + [e^1] + [e^1]} = .13447.$$

Moreover, if both players had proposed network G_1, then (because the state whose probability we are trying to compute contains *no* network proposed by any player) we would have

$$q(\omega_3|\omega_3, G_1, G_1) = 0.$$

Note that under the law of motion (14), states containing proposed networks that generate higher incremental payoffs to players relative to the status quo are assigned higher probabilities.

The Markov transition matrix induced by pure strategies $f_D^*(\cdot) = (f_{d_1}^*(\cdot), f_{d_2}^*(\cdot))$ is given by[6]

$$Q(f_D^*) = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .13447 & .13447 & 0 & 0 & .36553 & .36553 \\ 0 & 0 & .5 & .5 & 0 & 0 \\ 0 & 0 & .059601 & .059601 & .4404 & .4404 \\ .4404 & .4404 & 0 & 0 & .059601 & .059601 \\ 0 & 0 & 0 & 0 & .5 & .5 \end{pmatrix}.$$

Player $d_j's$ value function, $w_{d_j}^*(\cdot)$, in vector form, is given by

$$w_{d_j}^* = \left[I - \beta_{d_j} Q(f_D^*)\right]^{-1} v_{d_j}.$$

Here, $w_{d_j}^* \in R^6$ is the column vector listing the state-contingent values to player d_j of following strategy $f_{d_j}^*(\cdot)$, $v_{d_j} \in R^6$ is the column vector of state-contingent payoffs to player d_j (see expression (13)), and I is the 6×6 identity matrix. As noted already by Herings and Peeters (2004), because $Q(f_D^*)$ is a stochastic matrix, with rows nonnegative and summing to 1, it follows from Hadamard's Theorem that the inverse $\left[I - \beta_{d_j} Q(f_D^*)\right]^{-1}$ exists and is given by

$$\left[I - \beta_{d_j} Q(f_D^*)\right]^{-1} = \sum_{n=0}^{\infty} \beta_{d_j}^n Q(f_D^*)^n.$$

By computation, the value functions $w_D^*(\cdot) = (w_{d_1}^*(\cdot), w_{d_2}^*(\cdot))$, in vector form, are given by

$$w_{d_1}^* = \begin{pmatrix} 3.1559 & 3.0822 & 2.1041 & 2.0609 & 1.1439 & 1.0549 \end{pmatrix},$$

and

$$w_{d_2}^* = \begin{pmatrix} 2.0839 & 2.1134 & 1.0432 & 1.1146 & 3.0887 & 3.1244 \end{pmatrix}.$$

[6] The computations leading to transition matrix $Q(f_D)$ are quite long and tedious. They have been gathered in a working paper which is available upon request.

Given that

$$w^*_{d_j} = \left[I - \beta_{d_j} Q(f^*_D)\right]^{-1} v_{d_j},$$

we have for all states ω_i and players d_j that

$$w^*_{d_j}(\omega_i) - v_{d_j}(\omega_i) - \beta_{d_j} \sum_{k \in H} w^*_{d_j}(\omega_k) q(\omega_k|\omega_i, f^*_D(\omega_i)).$$

$$= w^*_{d_j}(\omega_i) - r_{d_j}(\omega_i, f^*_D(\omega_i)) - \beta_{d_j} \sum_{k \in H} w^*_{d_j}(\omega_k) q(\omega_k|\omega_i, f^*_D(\omega_i))$$

$$= 0,$$

Thus, $(w^*_D(\cdot), f^*_D(\cdot))$ solves the minimization problem

$$\min \sum_{d \in D} \sum_{i \in H} \left[w_d(\omega_i) - r_d(\omega_i, \sigma_D(\omega_i)) - \beta_d \sum_{j \in H} w_d(\omega_k) q(\omega_i|\omega_i, \sigma_D(\omega_i)) \right],$$

by attaining the lower bound of zero.

By Theorem 2, therefore, in order to show that the valuation function-strategy pair $(w^*_D(\cdot), f^*_D(\cdot))$ is an equilibrium, it remains only to show that no player d_j, in any state ω_i, can make himself better off by deviating to a *feasible proposal* $G^{'}$ $\in \Phi_{d_j}(\omega_i)$ from the proposal $f^*_{d_j}(\omega_i) \in \Phi_{d_j}(\omega_i)$ specified by his strategy $f^*_{d_j}(\cdot)$, assuming that all other players continue to follow their strategies, $f^*_{-d_j}(\cdot)$. This requires that for all states ω_i we recompute the *ith* row of the transition matrix $Q(f^*_D)$ (i.e., the row corresponding to state ω_i), given by

$$q(\cdot|\omega_i, G^{'}, f^*_{-d_j}(\omega_i)) := (q(\omega_1|\omega_i, G^{'}, f^*_{-d_j}(\omega_i)), \ldots, q(\omega_6|\omega_i, G^{'}, f^*_{-d_j}(\omega_i)),$$

for all feasible deviations $G^{'} \in \Phi_{d_j}(\omega_i)$ for all players d_j and then check that inequality

$$w^*_{d_j}(\omega_i) \geq v_{d_j}(\omega_i) + \beta_{d_j} \sum_{k \in H} w^*_{d_j}(\omega_k) q(\omega_k|\omega_i, G^{'}, f^*_{-d_j}(\omega_i)).$$

holds. For example, for the *feasible* state ω_6 player $d_2's$ deviation from $f^*_{d_2}(\omega_6) = G_3$ to G_2, we have

$$\begin{array}{c} q(\cdot|\omega_6, f^*_{d_1}(\omega_6), f^*_{d_2}(\omega_6)) := \begin{pmatrix} 0 & 0 & 0 & 0 & .5 & .5 \end{pmatrix} \\ \downarrow \\ q(\cdot|\omega_6, f^*_{d_1}(\omega_6), G_2,) := \begin{pmatrix} 0 & 0 & .059603 & .059603 & .4404 & .4404 \end{pmatrix} \end{array}$$

and given that $w^*_{d_2}(\omega_6) = 3.1244$, $v_{d_2}(\omega_6) = 3$, and $\beta_{d_2} = .04$, we conclude that

$w^*_{d_2}(\omega_6) = 3.1244$

$$> 3 + (.04)\begin{pmatrix} 0 & 0 & .059603 & .059603 & .4404 & .4404 \end{pmatrix}\begin{pmatrix} 2.0839 \\ 2.1134 \\ 1.0432 \\ 1.1146 \\ 3.0887 \\ 3.1244 \end{pmatrix} = 3.1146$$

$= v_{d_2}(\omega_6) + \beta_{d_2} \sum_{k \in H} w^*_{d_2}(\omega_k) q(\omega_k|\omega_6, f^*_{d_1}(\omega_6), G_2).$

Thus, player d_2 is not made better off by this deviation. Checking *all* such deviations for *all* players in *all* states, we conclude from Theorem 2 that $(w^*_D(\cdot), f^*_D(\cdot))$ is an equilibrium, and in particular, that the pure stationary strategy $f^*_D(\cdot)$ specified in Table 3 is Nash.

4 Endogenous Network Dynamics

4.1 The Equilibrium Markov Transition Kernel

Under stationary equilibrium, $\sigma^*_D(\cdot) = (\sigma^*_d(\cdot|\cdot))_{d \in D}$, the endogenous Markov process of network and coalition formation,

$$\{W^*_n\}_n = \{(G^*_n, S^*_n)\}_{n=0}^{\infty},$$

is governed by the equilibrium Markov transition kernel,

$$p^*(E|\omega) = \sum_{\omega' \in E} q(\omega'|\omega, \sigma^*_D(\omega))$$

$$= \sum_{G'_D \in \Phi(\omega)} \sum_{\omega' \in E} q(\omega'|\omega, G'_D)\sigma^*_D(G'_D|\omega)$$

$$= \sum_{G'_D \in \Phi(\omega)} q(E|\omega, G'_D)\sigma^*_D(G'_D|\omega).$$

for all $E \subseteq \Omega$.[7] Thus, for all $n = 0, 1, 2, \ldots$, the probability that the process $\{W^*_n\}_{n=0}^{\infty}$ reaches the set of states $E \in B(\Omega)$ in n moves given initial state $W_0 = \omega_0$ and history $W_1 = \omega_1, \ldots, W_{n-1} = \omega_{n-1}$ is given by

$$\Pi\left\{W^*_n \in E|W^*_{n-1} = \omega\right\} = p^*(E|\omega),$$

and thus depends only on state of the process after $n-1$ moves and not on the whole history of the process.[8] Moreover, for all $n = 0, 1, 2, \ldots$,

$$\Pi\{W^*_n \in E|W^*_0 = \omega\} = p^{*(n)}(E|\omega) = q^{(n)}(E|\omega, \sigma^*_D(\omega)),$$

[7] To put our model on a sound foundation, we will assume - as is standard in Markov Process Theory - that there is a probability space $(\Theta, \Im, \Pi)$ underlying our equilibrium process $\{W^*_n\}_n$.

[8] Here, Π is the probability measure defined on the measurable space $(\Theta, \Im)$ underlying the process of network and coalition formation. Thus, we are assuming that the Markov process $\{W^*_n\}_n$ of

where the n-step transition $p^{*(n)}(\cdot|\cdot)$ is given recursively as follows: for all $\omega \in \Omega$ and $E \subseteq \Omega$,

$$p^{*(n)}(E|\omega) = \sum_{\omega' \in \Omega} p^*(E|\omega')p^{*(n-1)}(\omega'|\omega) = \sum_{\omega' \in \Omega} p^{*(n-1)}(E|\omega')p^*(\omega'|\omega). \tag{15}$$

4.2 The Equilibrium Markov Transition Matrix

We can also represent the equilibrium Markov transition $p^*(\cdot|\cdot)$ via a Markov transition matrix. In particular, given stationary equilibrium, $\sigma_D^*(\cdot) = (\sigma_d^*(\cdot|\cdot))_{d \in D}$, let Q^* be the resulting $N \times N$ *equilibrium* Markov transition matrix, where Q^* has typical entry q_{ij}^* $(= q(\omega_j|\omega_i, \sigma_D^*(\omega_i)))$, where q_{ij}^* is the probability that nature moves from state $\omega_i = (G_i, S_i)$ to state $\omega_j = (G_j, S_j)$ given stationary equilibrium proposal strategies, $\sigma_D^*(\cdot)$. Note that each row of Q^*, given by

$$p_i^* = (q_{i1}^*, q_{i2}^*, \ldots, q_{iN}^*) := (Q^*)_i, \tag{16}$$

is a conditional probability measure on the state space Ω, that is, $p_i^* \in \mathcal{P}(\Omega)$, where the set of probability measures $\mathcal{P}(\Omega)$ is given by

$$\mathcal{P}(\Omega) = \left\{ p = (q_1, \ldots, q_N) \in R^N : q_j \geq 0 \text{ and } \sum_{j \in H} q_j = 1 \right\}. \tag{17}$$

Given initial probability measure $\gamma^* = (\gamma_1^*, \gamma_2^*, \ldots, \gamma_N^*) \in \mathcal{P}(\Omega)$ prescribing the probability with which the initial state i_0 (or equivalently, $\omega_{i_0} = (G_{i_0}, S_{i_0})$) is chosen, the probability that the process is in state i_n after n moves is given by

$$\Pi\{W_n^* = \omega_{i_n}\} = (\gamma^* Q^{*n})_{i_n} = \sum_{i_0 \in H} \gamma_{i_0}^* (Q^{*n})_{i_0 i_n}$$

where

Q^{*n} is the matrix obtained by multiplying $Q^*$by itself n times,

$(\gamma^* Q^{*n})_{i_n}$ is the i_n^{th} component of the row vector $\gamma^* Q^{*n}$,
and
$(Q^{*n})_{i_0 i_n} := q_{i_0 i_n}^{*(n)}$ is the $(i_0, i_n)^{th}$ entry of the matrix Q^{*n}.

For example if $n = 2$, then

$$(\gamma^* Q^{*n})_{i_2} = \sum_{i_0 \in H} \gamma_{i_0}^* (Q^{*2})_{i_0 i_2} = \sum_{i_0 \in H} \gamma_{i_0}^* (\sum_{i_1 \in H} q_{i_0 i_1}^* q_{i_1 i_2}^*).$$

---

network and coalition formation is a sequence of Ω-valued, $\Im$-measurable functions

$$W_n^* : \Theta \to \Omega := (\mathbb{G} \times \mathcal{F}),$$

such that,

$$\begin{gathered}\Pi\{W_n \in E | W_0 = \omega_0, W_1 = \omega_1, \ldots, W_{n-1} = \omega_{n-1}\} \\ = \Pi\{W_n \in E | W_{n-1} = \omega_{n-1}\} \\ \Pi_{\omega_{n-1}}\{W_n \in E\}.\end{gathered}$$

The probability that the process is in state i_n after n moves starting from state i_0 is given by

$$\Pi_{\omega_{i_0}}\{W_n^* = \omega_{i_n}\} := \Pi\{W_n^* = \omega_{i_n}|W_0^* = \omega_{i_0}\} = (Q^{*n})_{i_0 i_n} = q^{*(n)}_{i_0 i_n}$$

For example if $n = 2$, then

$$(Q^{*2})_{i_0 i_2} = q^{*(2)}_{i_0 i_2} = \sum_{i_1 \in H} q^*_{i_0 i_1} q^*_{i_1 i_2}.$$

4.3 The Equilibrium Markov Supernetwork

Corresponding to the equilibrium Markov transition matrix Q^* there is a unique directed network M^* - a supernetwork (Page, Wooders, and Kamat (2005) - where

$$M^* \subset [0,1] \times (\Omega \times \Omega),$$

with typical connection $(q^*_{ij}, (\omega_i, \omega_j))$ where q^*_{ij} is the ij^{th} entry in the equilibrium Markov transition matrix Q^* and ω_i and ω_j are network-coalition pairs contained in the state space. The connection $(q^*_{ij}, (\omega_i, \omega_j)) \in M^*$ is *active* if and only if the process of network-coalition formation $\{W_n^*\}$ governed by equilibrium Markov transition Q^* is such that for all $n = 1, 2, \ldots,$

$$\Pi\{W_n^* = \omega_j | W_{n-1}^* = \omega_i\} = q^*_{ij} > 0.$$

Examples 4: Endogenous Supernetworks

(1) Consider process of network-coalition formation $\{W_n^*\}$ with state space

$$\Omega := \{\omega_1, \omega_2, \ldots, \omega_6\},$$

governed by equilibrium Markov transition matrix Q^* given by

$$Q^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ .4 & .6 & 0 & 0 & 0 & 0 \\ .3 & 0 & .4 & .2 & .1 & 0 \\ 0 & 0 & 0 & .3 & .7 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & .8 & 0 & .2 \end{pmatrix}.$$

The corresponding equilibrium Markov supernetwork M^* (with only active connections shown) is depicted in Figure 4.

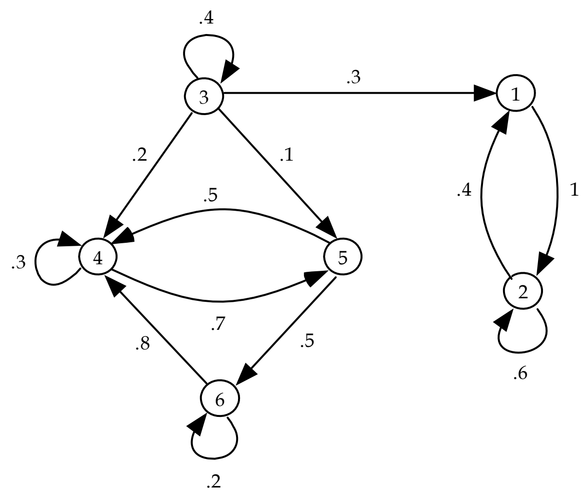

Figure 4: The Equilibrium Markov Supernetwork

(2) Consider the process of network-coalition formation $\{W_n(f_D^*)\}$ governed by the equilibrium Markov transition $Q(f_D^*)$ induced by the pure stationary equilibrium strategies $f_D^* = (f_{d_1}^*, f_{d_2}^*)$ in Example 3, and recall that the state space is given by

$$\Omega = \left\{ \underbrace{(G_1,\{d_1\})}_{\omega_1}, \underbrace{(G_1,\{d_2\})}_{\omega_2}, \underbrace{(G_2,\{d_1\})}_{\omega_3}, \underbrace{(G_2,\{d_2\})}_{\omega_4}, \underbrace{(G_3,\{d_1\})}_{\omega_5}, \underbrace{(G_3,\{d_2\})}_{\omega_6} \right\}.$$

The equilibrium Markov transition matrix, $Q(f_D^*)$, is given by

$$Q(f_D^*) = \begin{pmatrix} .5 & .5 & 0 & 0 & 0 & 0 \\ .13447 & .13447 & 0 & 0 & .36553 & .36553 \\ 0 & 0 & .5 & .5 & 0 & 0 \\ 0 & 0 & .059601 & .059601 & .4404 & .4404 \\ .4404 & .4404 & 0 & 0 & .059601 & .059601 \\ 0 & 0 & 0 & 0 & .5 & .5 \end{pmatrix}.$$

The corresponding equilibrium Markov supernetwork $M(f_D^*)$ (with only active connections shown) is depicted in Figure 5.

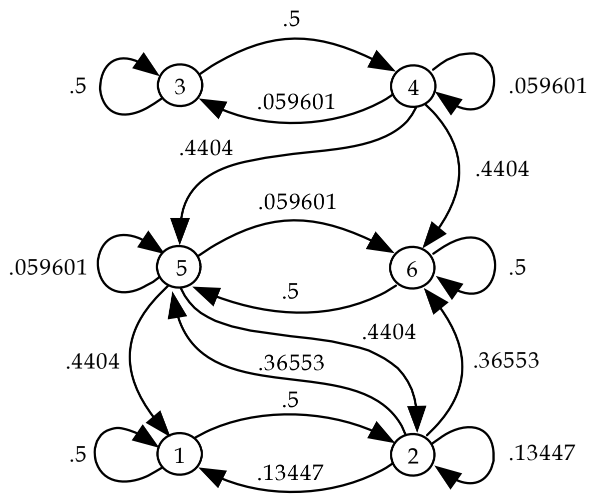

Figure 5: The Equilibrium Markov Supernetwork $M(f_D^*)$

4.4 Three Fundamental Results on Finite-State Markov Chains

4.4.1 Hitting Times

Often we will be interested in determining the probability with which the network-coalition formation process reaches (*hits in finite time*) a particular network-coalition pair $\omega_j = (G_j, S_j)$ *after* leaving a particular network-coalition pair $\omega_i = (G_i, S_i)$.

To begin, let the integer-valued random variable, $T_{\omega_j}^*(\cdot) : \Theta \to \{0, 1, 2, \ldots\}$, given by

$$T_{\omega_j}^* := \min\{n \geq 1 : W_n^* = \omega_j\} \tag{18}$$

be the *hitting time* of network-coalition formation process $\{W_n^*\}_n$ for state $\omega_j \in \Omega$, and let

$$\rho_{ij}^* := \Pi\left\{T_{\omega_j}^* < \infty | W_0^* = \omega_i\right\}. \tag{19}$$

be the *probability* that the process $\{W_n^*\}_n$ reaches the state ω_j after leaving state ω_i at time zero in finite time. Also, the *expected hitting time* (expected number of moves) for the process $\{W_n^*\}_n$ to reach ω_i again after leaving ω_i at time zero is

$$E_{\omega_i} T_{\omega_i}^* := E(T_{\omega_i}^* | W_0^* = \omega_i) = \sum_{n=0}^{\infty} \Pi\left\{T_{\omega_i}^* > n | W_0^* = \omega_i\right\} = \sum_{n=0}^{\infty} \Pi_{\omega_i}\left\{T_{\omega_i}^* > n\right\}. \tag{20}$$

Next consider the sequence of hitting times $\{T_{\omega_i}^{*k}\}_{k=0}^{\infty}$ defined recursively as follows: $T_{\omega_i}^{*0} := 0$ and for $k \geq 1$,

$$T_{\omega_i}^{*k} := \inf\left\{n > T_{\omega_i}^{*k-1} : W_n^* = \omega_i\right\}. \tag{21}$$

$T^{*k}_{\omega_i}$ is the number of moves required for the k^{th} return to ω_i. Note that $T^{*1}_{\omega_i} > 0$, so any visit to ω_i at time 0 does not count. Note also that $T^{*}_{\omega_i} = T^{*1}_{\omega_i}$. The following two results on hitting times can be found in Durrett (2005) section 5.3.

Theorem 3 (*On Hitting Times*)

Let $\{W^*_n\}_{n=0}^{\infty}$ be an endogenous Markov process of network and coalition formation governed by equilibrium Markov transition Q^* with state space

$$\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\}.$$

The following statements are true:

(1) For all k and all states ω_i and ω_j, $\Pi\left\{T^{*k}_{\omega_j} < \infty | W^*_0 = \omega_i\right\} = \rho^*_{ij}\rho^{*k-1}_{jj}$.

(2) For all states ω_i, ω_j, and ω_h, $\rho^*_{ih} \geq \rho^*_{ij}\rho^*_{jh}$.

4.4.2 Recurrence and Transience

A network-coalition pair $\omega_i = (G_i, S_i)$ is said to be *recurrent* if $\rho^*_{ii} = 1$ and *transient* if $\rho^*_{ii} < 1$. By part (1) of Theorem 3, if ω_j is recurrent, then the number of moves required for the k^{th} return to ω_j is finite with probability 1 - or stated formally, if ω_j is recurrent, then

$$\Pi\left\{T^{*k}_{\omega_j} < \infty | W^*_0 = \omega_j\right\} = 1 \text{ for all } k.$$

A network-coalition pair $\omega_i = (G_i, S_i)$ is said to be *positive recurrent* if $E_{\omega_i}T^*_{\omega_i} < \infty$ and null recurrent if $E_{\omega_i}T^*_{\omega_i} = \infty$. It is easy to show that all recurrent states of a finite state Markov chain are positive recurrent.

Given any state $\omega_j \in \Omega$, the number of *visitations* to ω_j by the process $\{W^*_n\}_n = \{(G^*_n, S^*_n)\}_{n=1}^{\infty}$ after time zero is given by

$$V^*_{\omega_j} := \sum_{n=1}^{\infty} I_{\{W^*_n = \omega_j\}}. \tag{22}$$

If network-coalition pair $\omega_j = (G_j, S_j)$ is transient, then the expected number of visitations to ω_j starting from network-coalition pair $\omega_i = (G_i, S_i)$ is given by

$$\left.\begin{aligned}
&g^*_{ij} := E_{\omega_i}[V^*_{\omega_j}] = \textstyle\sum_{k=1}^{\infty} \Pi_{\omega_i}\{V^*_{\omega_j} \geq k\} \\
&= \textstyle\sum_{k=1}^{\infty} \Pi\left\{T^{*k}_{\omega_j} < \infty | W^*_0 = \omega_i\right\} \\
&= \textstyle\sum_{k=1}^{\infty} \rho^*_{ij}\rho^{*k-1}_{jj} \text{ (by Theorem 3 (1))} \\
&= \tfrac{\rho^*_{ij}}{1-\rho^*_{jj}} < \infty.
\end{aligned}\right\} \tag{23}$$

We can conclude from (23) that in fact ω_j is recurrent if and only if

$$g^*_{jj} := E_{\omega_j}[V^*_{\omega_j}] = \infty.$$

The following classical result on recurrent states can also be found in Durrett (2005), for example.

Theorem 4 (*Recurrence is Contagious*)

Let $\{W^*_n\}_{n=0}^{\infty}$ be an endogenous Markov process of network and coalition formation governed by equilibrium Markov transition Q^* with state space

$$\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\}.$$

If network-coalition pair $\omega_i = (G_i, S_i)$ is recurrent and network-coalition pair $\omega_j = (G_j, S_j)$ is reachable, that is, if $\rho^*_{ij} > 0$, then $\omega_j = (G_j, S_j)$ is recurrent and $\rho^*_{ji} = 1$.

The following classical results tell us the precise relationship between irreducibility, recurrence, and closedness (see, for example, Durrett (2005)).

Theorem 5 (*Closedness and Irreducibility Imply Recurrence*)

Let $\{W^*_n\}_{n=0}^{\infty}$ be an endogenous Markov process of network and coalition formation governed by equilibrium Markov transition Q^* with state space

$$\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\}.$$

The following statements are true:

(1) If $A^* \subseteq \Omega$ is closed, then it contains at least one recurrent network-coalition pair $\omega_i = (G_i, S_i)$, that is, $\rho^*_{ii} = 1$ (or equivalently, $g^*_{ii} = \infty$) for some $\omega_i \in A^*$. Moreover, $A^* \subseteq \Omega$ is closed if and only if for all $\omega_i \in A^*$ and for all n

$$\Pi\{W^*_n \in A^* | W^*_0 = \omega_i\} = 1. \ (*)$$

(2) If $A^* \subseteq \Omega$ is closed and irreducible, then it is recurrent.

Proof. : (1) The proof of (*) is straightforward. Now suppose A^* is closed but contains no recurrent states, so that $g^*_{jj} < \infty$ for all $\omega_j \in A^*$. But now we have a contradiction because by (23) and (*) in part (1) we have for all $\omega_i \in A^*$ and for all n

$$\infty > \sum_{j \in H_{A^*}} g^*_{ij} := \sum_{\omega_j \in A^*} E^*_{\omega_i}[V^*_{\omega_j}]$$

$$= \sum_{j \in H_{A^*}} \sum_{n=1}^{\infty} q^{*(n)}_{ij} = \sum_{n=1}^{\infty} \sum_{j \in H_{A^*}} q^{*(n)}_{ij}$$

$$= \sum_{n=1}^{\infty} \Pi\{W^*_n \in A^* | W^*_0 = \omega_i\}$$

$$= \lim_{n \to \infty} n.$$

(2) Suppose A^* is closed and irreducible. By closedness we know by part (1) that there is at least one recurrent state in A^*. By the Contagious Theorem 3 we know that any state reachable from this recurrent state is also recurrent. By irreducibility, we know that all states in A^* are reachable. Therefore, all states in A^* are recurrent. ■

4.5 Decomposition and Ergodicity

One of our primary objectives is to show that an endogenous Markov process of network and coalition formation governed by equilibrium Markov transition Q^* with state space

$$\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\},$$

generates a *unique* decomposition of the state space Ω of network-coalition pairs given by

$$\Omega = \cup_k A^{*k} \cup T,$$

where each A_k is a basin of attraction and T is transient. This decomposition is the *unique stability signature of the Nash equilibrium* $(\sigma_d^*(\cdot|\cdot))_{d \in D}$ of the dynamic game of network formation. We will also show that this endogenous network dynamic possesses a unique finite set of ergodic probability measures (i.e., long run equilibrium probability measures) over network-coalition pairs, one for each basin of attraction, and that each invariant probability measure is then a convex combination of these ergodic measures. This set of ergodic measures is the *unique probabilistic signature of the Nash equilibrium.*

4.5.1 Basins of Attraction

We say that there is a *path* from state ω_i to state ω_j if $\rho_{ij}^* > 0$. If, in addition, $\rho_{ji}^* > 0$, so there is a path back, then we say that state ω_i and ω_j are on the same circuit. In particular, if $\rho_{ii}^* > 0$, then there is a path from ω_i to ω_i.

We say that a set of states $A^* \subseteq \Omega$ is *irreducible* if for each pair of states ω_i and ω_j contained in A^*, there is a path from ω_i to ω_j. Thus, A^* is irreducible if and only if for every pair of states ω_i and ω_j contained in A^*, $\rho_{ij}^* > 0$ and $\rho_{ji}^* > 0$ - and thus, if A^* is irreducible, then $\rho_{ii}^* > 0$ for all $\omega_i \in A^*$.[9] In fact, by the Contagious Theorem 3, if A^* is irreducible, then *all* states in A^* are either recurrent ($\rho_{ii}^* = 1$ for all $\omega_i \in A^*$) or transient ($\rho_{ii}^* < 1$ for all $\omega_i \in A^*$) . Finally, we say that a set of states $A^* \subseteq \Omega$ is *closed* if for all $\omega_i \in A^*$, $\rho_{ij}^* > 0$ implies that $\omega_j \in A^*$.

Definition 5 (*Basins of Attraction*)

A set of states $A^* \subseteq \Omega$ is said to be a basin of attraction for the process $\{W_n^*\}_n$ governed by Markov transition Q^* if A^* is closed and irreducible.

[9] If the entire state space Ω is irreducible, we say that the process $\{W_n^*\}_n$ governed by Markov transition Q^* is *irreducible.*

Note that by Theorem 5(2), each state contained in a basin of attraction is recurrent, that is,

$$\rho^*_{ii} := \Pi\left\{T^*_{\omega_i} < \infty | W^*_0 = \omega_i\right\} = 1.$$

Thus, all basins of attraction generated the process $\{W^*_n\}_n$ on *finite* state space Ω are closed, irreducible, and positive recurrent.

4.5.2 The Existence of a Unique Decomposition into Basins of Attraction

We begin with an observation essentially due to Durrett (2005): Any Markov network-coalition formation process, say $\{W^*_n\}_n$, on the *finite* state space

$$\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\},$$

is such that starting with any network-coalition pair $\omega = (G, S)$, there is another network-coalition pair $\omega' = (G', S')$ such that either **(a)** $\rho^*_{\omega\omega'} > 0$ and $\rho^*_{\omega'\omega} = 0$ (there is an active path from ω to ω', but not back), in which case it follows from Theorem 4 that ω is transient (otherwise, we would have $\rho^*_{\omega'\omega} = 1$) or **(b)** $\rho^*_{\omega\omega'} > 0$ implies that $\rho^*_{\omega'\omega} > 0$ (there is an active path from ω to ω', and back), in which case the set of network-coalition pairs

$$\Omega_\omega := \left\{\omega' \in \Omega : \rho^*_{\omega\omega'} > 0\right\}$$

is closed and irreducible (i.e., a basin of attraction). First, to see that (b) implies that Ω_ω is irreducible observe that if ω' and ω'' are in Ω_ω, then by Theorem 3(2) $\rho^*_{\omega'\omega''} \geq \rho^*_{\omega'\omega}\rho^*_{\omega\omega''} > 0$. Hence ω'' can be reached from ω'. Second, to see that (b) implies that Ω_ω is closed observe that if $\widetilde{\omega} \in \Omega$ and $\rho^*_{\omega'\widetilde{\omega}} > 0$, then by Theorem 3(2), $\rho^*_{\omega\widetilde{\omega}} \geq \rho^*_{\omega\omega'}\rho^*_{\omega'\widetilde{\omega}} > 0$. Hence, $\widetilde{\omega} \in \Omega_\omega$.

Our main result on the existence of a unique decomposition of the state space into basins of attraction is as follows:

Theorem 6 (*The Existence of a Unique Decomposition into Basins of Attraction*)

Let $\{W^*_n\}^\infty_{n=0}$ be an endogenous Markov process of network and coalition formation governed by equilibrium Markov transition Q^* with state space

$$\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\}.$$

The following statements are true:

(1) There is a unique decomposition of the state space Ω of network-coalition pairs given by

$$\Omega = \cup_k A^{*k} \cup T,$$

where each A^{*k} is a basin of attraction and T is transient. Moreover, $\cup_k A^{*k}$ is the unique partition of the set of recurrent network-coalition pairs, $R^* := \{\omega \in \Omega : \rho^*_{\omega\omega} = 1\}$ into closed recurrent sets. Thus, $R^* = \cup_k A^{*k}$.

(2) *Starting from any network-coalition pair* $\omega \in \Omega$, *the process* $\{W_n^*\}_{n=0}^{\infty}$ *reaches a network-coalition pair* ω' *contained in a basin of attraction in finite time with probability* 1. *Thus, for all* $\omega \in \Omega$, *the exists an integer* n_ω *such that*

$$\Pi\left\{W_n^* \in \left[\cup_k A^{*k}\right] | W_0^* = \omega\right\} = 1 \textit{for all } n \geq n_\omega.$$

Proof. (1) By Theorem 5(1), because Ω is finite, the set of recurrent network-coalition pairs R^* is nonempty. Consider the sets

$$\Omega_\omega := \left\{\omega' \in \Omega : \rho^*_{\omega\omega'} > 0\right\},$$

for $\omega \in R^*$. By Theorem 4, $\Omega_\omega \subseteq R^*$ and if $\omega' \in \Omega_\omega$, then $\rho^*_{\omega'\omega} = 1$. Thus, by the discussion before the statement of Theorem 6, for each $\omega \in R^*$, Ω_ω is closed and irreducible, and therefor by Theorem 5(2) recurrent. Because the sets Ω_ω, $\omega \in R^*$ are equivalence classes of recurrent network-coalition pairs, for all ω and ω' in R^*, either $\Omega_\omega \cap \Omega_{\omega'} = \emptyset$ or $\Omega_\omega = \Omega_{\omega'}$, and $\cup_{\omega \in R^*}\Omega_\omega = R^*$. Let $\left\{\Omega_{\omega_{i_k}}\right\}_k$ be the unique finite partition of R^* into recurrent classes and note that the set $\Omega \backslash \left[\cup_k \Omega_{\omega_{i_k}}\right]$ is transient. Letting $\Omega_{\omega_{i_k}} = A^{*k}$ for all k, the proof of (1) is complete.

(2) Recalling that $T := \Omega \backslash \left[\cup_k A^{*k}\right]$ is transient, note that

$$\begin{aligned}
&\Pi\left\{W_n^* \in T \text{ for all } n\right\} \\
&\leq \textstyle\sum_{\omega' \in T} \Pi\left\{W_n^* = \omega' \text{ for infinitely many } n\right\} = 0 \\
&\leq \textstyle\sum_{\omega' \in T}\left[\sum_{\omega \in \Omega} \Pi\left\{W_n^* = \omega' \text{ for infinitely many } n | W_0^* = \omega\right\} \Pi\left\{W_0^* = \omega\right\}\right] \\
&= 0
\end{aligned}$$

Thus, depending the starting state $\omega \in T$, at some finite time point n_ω the process leaves T, enters $\left[\cup_k A^{*k}\right]$ and remains there. ■

Examples 5: Basins of Attraction

(1) In example 4(1), by direct observation of Figure 4, we can conclude that sets $A^{*1} = \{\omega_4, \omega_5, \omega_6\}$ and $A^{*2} = \{\omega_1, \omega_2\}$ are closed and irreducible, and hence by Theorem 5(2) recurrent. Thus, A^{*1} and A^{*2} are the basins of attraction generated by the network-coalition formation process $\{W_n^*\}$ governed by the equilibrium Markov transition Q^*.

(2) In example 4(2), by direct observation of Figure 5, we can conclude that the set $A^* = \{\omega_1, \omega_2, \omega_5, \omega_6\}$ is closed and irreducible (and hence also recurrent). Thus, A^* is the basin of attraction generated by the network-coalition formation process $\{W_n(f_D^*)\}$ governed by the equilibrium Markov transition $Q(f_D^*)$ induced by the pure stationary equilibrium strategies $f_D^* = (f_{d_1}^*, f_{d_2}^*)$ in Example 3.

4.5.3 Invariant and Ergodic Probability Measures

A probability measure $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathcal{P}(\Omega)$ on the state space of feasible network-coalition pairs

$$\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\}.$$

is invariant for Markov transition Q^* (i.e., is Q^*-invariant) if

$$\lambda Q^* = \sum_{i \in H} \lambda_i (Q^*)_i = \sum_{i \in H} \lambda_i p_i^* \ = \lambda. \tag{24}$$

Thus, if probability measure λ is Q^*-invariant, then for any set of network-coalition pairs $E \subseteq \Omega$, if the current status quo network-coalition pair $\omega_{i_n} = (G_{i_n}, S_{i_n})$ is chosen according to probability measure λ - so that the probability that ω_{i_n} lies in E is $\lambda(E) := \sum_{i_n \in H_E} \lambda_{i_n}$ - then the probability that any future period's network-coalition pair $\omega_{i_{n+m}} = (G_{i_{n+m}}, S_{i_{n+m}})$ lies in E is also $\lambda(E) := \sum_{i_{n+m} \in H_E} \lambda_{i_{n+m}}$. Denote by $\mathcal{I}^*$ the collection of all Q^*-invariant measure.

Let $\mathcal{A}^*$ denote the collection of all basins of attraction (i.e., all closed, irreducible sets). A Q^*-invariant measure λ is said to be Q^*-ergodic if $\lambda(A) = 0$ or $\lambda(A) = 1$ for all basins of attraction $A \in \mathcal{A}^*$. Denote by $\mathcal{E}^*$ the collection of all Q^*-ergodic measures. Because the Q^*-ergodic probability measures are the extreme points of the (possibly empty) convex set $\mathcal{I}^*$ of Q^*-invariant measures (see Theorem 19.25 in Aliprantis and Border (1999)), each measure λ in $\mathcal{I}^*$ can be written as a convex combination of the measures in $\mathcal{E}^*$.

4.5.4 The Existence of a Unique Set of Ergodic Probability Measures

The set of invariant probability measures $\mathcal{I}^*$ for equilibrium Markov transition Q^* with state space $\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\}$ is given by the set

$$\mathcal{I}^* = \{\lambda \in \mathcal{P}(\Omega) : \lambda Q^* = \lambda\}.$$

We will show that if

$$\left\{A^{*1}, A^{*2}, \ldots, A^{*L}\right\}$$

is the unique, finite, disjoint collection of basins of attraction generated by equilibrium Markov transition Q^*, then corresponding to each basin of attraction A^{*k} there is unique ergodic probability measure $\alpha^{*k}(\cdot)$ concentrated on A^{*k} such that $\alpha^{*k}(\omega) > 0$ for all $\omega \in A^{*k}$. Moreover, we will show that the set of all ergodic probability measures for transition Q^* is given by

$$\mathcal{E}^* = \{\alpha^{*1}(\cdot), \ldots, \alpha^{*L}(\cdot)\}.$$

Thus, we will conclude that each Q^*-invariant probability measure $\lambda \in \mathcal{I}^*$ is a convex combination of the ergodic measures in $\mathcal{E}^*$ (i.e., we will conclude that $\mathcal{I}^* = co\mathcal{E}^*$, where *co* denotes convex hull). Finally, we show how to compute these ergodic measures.

The following results are variations on classical results for finite state Markov chains (see for example Durrett (2005), Kemeny and Snell (1960), or Norris (1997)).

Theorem 7 (*Ergodicity and Invariance Results for Probability Measures and Functions for Finite State Markov Processes*)

Let Q^* be an equilibrium Markov transition on state space $\Omega := \{\omega_1, \omega_2, \ldots, \omega_N\}$ with basins of attraction

$$\left\{A^{*1}, A^{*2}, \ldots, A^{*L}\right\}.$$

The following statements are true:

(1) For each basin of attraction A^{*k} there is unique ergodic probability measure $\alpha^{*k}(\cdot)$ with support contained in A^{*k} such that $\alpha^{*k}(\omega) > 0$ for all $\omega \in A^{*k}$.

Moreover, for each $\omega \in A^{*k}$,

$$\alpha^{*k}(\omega) = \frac{1}{E_\omega T^*_\omega}.$$

(2) The set of all Q^*-ergodic probability measures is given by $\mathcal{E}^* = \{\alpha^{*1}(\cdot), \ldots, \alpha^{*L}(\cdot)\}$ and $\mathcal{I}^* = co\mathcal{E}^*$, where co denotes convex hull.

(3) For each basin of attraction A^{*k} and each initial state $\omega_{i_0} \in A^{*k}$

$$\Pi\left\{\lim_n \tfrac{1}{n}\textstyle\sum_{m=0}^{n-1} f(W^*_m) = f_{A^{*k}} | W^*_0 = \omega_{i_0}\right\} = 1,$$

and

$$\Pi\left\{\lim_n \tfrac{1}{n}\textstyle\sum_{m=0}^{n-1} I_{\left\{W^*_m = \omega_{i_0}\right\}} = \alpha^{*k}(\omega_{i_0}) | W^*_0 = \omega_{i_0}\right\} = 1.$$

Here,

$$f_{A^{*k}} := \sum_{\omega_j \in A^{*k}} f(\omega_j)\alpha^{*k}(\omega_j)$$

is the expected value of the function (random variable) $f(\cdot)$ on the basin of attraction A^{*k} with respect to the ergodic probability measure $\alpha^{*k}(\cdot)$ concentrated on A^{*k}, while the random variable

$$\frac{1}{n}\sum_{m=0}^{n-1} I_{\left\{W^*_m = \omega_{i_0}\right\}}$$

is the average amount of time (average number of moves) the processes spends in state $\omega_{i_0} \in A^{*k}$ before n.

Proof. (1) Suppose basin of attraction A^{*k} is given by

$$A^{*k} = \{\omega_{1_k}, \ldots, \omega_{N_k}\} \subseteq \Omega.$$

Let Q^*_k be the $N_k \times N_k$ submatrix of the Markov transition matrix Q^*. The matrix Q^*_k has typical entry $q^*_{i_k j_k} = q(\omega_{j_k} | \omega_{i_k}, \sigma^*_D(\omega_{i_k}))$ where $q^*_{i_k j_k}$ is the probability that nature moves from state $\omega_{i_k} = (G_{i_k}, S_{i_k})$ to state $\omega_{j_k} = (G_{j_k}, S_{j_k})$ given stationary equilibrium proposal strategies $\sigma_D^*(\cdot)$. Because the basin of attraction A^{*k} is closed and irreducible, Q_k^* is also a Markov transition matrix for the process confined to A^{*k}. Recall that if the process begins in A^{*k} it will stay in A^{*k}. In particular, by part (1) of Theorem 4 that for all $\omega_i \in A^{*k}$ and for all n,

$$\Pi\left\{W_n^* \in A^{*k} | W_0^* = \omega_i\right\} = 1.$$

Hence the process on A^{*k} is irreducible and every state in A^{*k} is positive recurrent. Thus, part (1) follows from Theorem 1.7.7 in Norris (1997).

(2) By Theorem 3.2.10 in Strook (2005), because Ω contains positive recurrent states (for example any state in a basin of attraction), the set of invariant probability measures $\mathcal{I}^*$ is nonempty and $\mathcal{I}^*$ is clearly convex and compact. Thus by the finite-dimensional Krein-Milman Theorem (Aliprantis and Border (2005), p 297) $\mathcal{I}^*$ is the convex hull of its extreme points. It only remains to show that each ergodic probability measure in $\mathcal{E}^* = \{\alpha^{*1}(\cdot), \ldots, \alpha^{*L}(\cdot)\}$ is an extreme point of $\mathcal{I}^*$ and that $\mathcal{E}^*$ contains all the extreme points of $\mathcal{I}^*$. But these conclusions are an immediate consequence of Theorem 3.2.10 in Strook (2005).

(3) Because the process on A^{*k} is irreducible and every state in A^{*k} is positive recurrent, part (3) follows from Theorem 1.10.2 in Norris (1997). ■

We conclude this section by showing how to compute the unique ergodic probability measure $\alpha^{*k}(\cdot)$ corresponding to any basin of attraction A^{*k}. First because each basin is closed and irreducible, consisting entirely of positive recurrent states, the process confined to A^{*k} and governed by Markov transition matrix Q_k^* is ergodic.

Here we follow the approach introduced in the classic book by Kemeny and Snell (1960) on finite Markov chains. To begin, let h be a vector of ones in R^{N_k}, that is, let

$$h_k = (1, \ldots, 1) \in R^{N_k}.$$

Also, let β be *any* probability vector in R^{N_k}, that is, let

$$\beta_k = (\beta_{1k}, \ldots, \beta_{N_k})$$

where $\beta_{j_k} \geq 0$ for all $j_k \in \{1_k, \ldots, N_k\}$ and $\sum_{j_k=1_k}^{N_k} \beta_{j_k} = 1$. Finally, let Z_{β_k} be the $N_k \times N_k$ matrix given by

$$Z_{\beta_k} = (I - Q_k^* + h_k \beta_k)^{-1}$$

where

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} \beta_{1k} & , \ldots, & \beta_{N_k} \end{pmatrix} = \begin{pmatrix} \beta_{1k} & , \ldots, & \beta_{N_k} \\ & \vdots & \\ \beta_{1k} & , \ldots, & \beta_{N_k} \end{pmatrix}_{N_k \times N_k}.$$

Then, the ergodic probability measure

$$\alpha^{*k} := (\alpha_{1_k}^{*k}, \ldots, \alpha_{N_k}^{*k}) := (\alpha^{*k}(\omega_{1_k}), \ldots, \alpha^{*k}(\omega_{N_k}))$$

is given by

$$\beta_k Z_{\beta_k} = \alpha^{*k}.$$

What is interesting is that $\beta_k Z_{\beta_k} = \alpha^{*k}$ for all probability vectors β_k.

Returning to Example 1 above, recall that the Markov transition

$$Q^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ .4 & .6 & 0 & 0 & 0 & 0 \\ .3 & 0 & .4 & .2 & .1 & 0 \\ 0 & 0 & 0 & .3 & .7 & 0 \\ 0 & 0 & 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & .8 & 0 & .2 \end{pmatrix}$$

generates two basins of attraction, $A^{*1} = \{\omega_1, \omega_2\}$ and $A^{*2} = \{\omega_4, \omega_5, \omega_6\}$. The state $T = \{\omega_3\}$ is transient.

First, we will compute $\alpha^{*1} := (\alpha^{*1}_{1_k}, \alpha^{*1}_{2_k}) := (\alpha^{*1}(\omega_{1_k}), \alpha^{*1}(\omega_{2_k}))$. Arbitrarily choosing vector, $\beta_1 = (1, 0)$,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \vdots & \\ 1 & 0 \end{pmatrix}_{2_1 \times 2_1}.$$

We have

$$Z_{\beta_1} = (I - Q_1^* + h_1 \beta_1)^{-1}$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ .4 & .6 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} .28571 & .71429 \\ -.42857 & 1.4286 \end{pmatrix}.$$

Therefore,

$$\beta_1 Z_{\beta_1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} .28571 & .71429 \\ -.42857 & 1.4286 \end{pmatrix}$$

$$= \begin{pmatrix} .28571 & .71429 \end{pmatrix}$$

$$= (\alpha^{*1}_{1_1}, \alpha^{*1}_{2_1}).$$

Checking, we have

$$\alpha^{*1} Q_1^*$$

$$= \begin{pmatrix} .28571 & .71429 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ .4 & .6 \end{pmatrix}$$

$$= \begin{pmatrix} .28572 & .71428 \end{pmatrix}.$$

Second, computing $\alpha^{*2} := (\alpha^{*2}_{1_k}, \alpha^{*2}_{2_k}, \alpha^{*2}_{3_k})$, we arbitrarily choosing $\beta_2 = (0, 1, 0)$,

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ & \vdots & \\ 0 & 1 & 0 \end{pmatrix}_{3_2 \times 3_2} .$$

We have

$$Z_{\beta_2} = (I - Q_2^* + h_2\beta_2)^{-1}$$

$$= \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} .3 & .7 & 0 \\ .5 & 0 & .5 \\ .8 & 0 & .2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} 1.2281 & -.14035 & -.087719 \\ .46784 & .32749 & .20468 \\ .64327 & -.54971 & .90643 \end{pmatrix}.$$

Therefore,

$$\beta_2 Z_{\beta_2} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1.2281 & -.14035 & -.087719 \\ .46784 & .32749 & .20468 \\ .64327 & -.54971 & .90643 \end{pmatrix}$$

$$= \begin{pmatrix} .46784 & .32749 & .20468 \end{pmatrix}$$

$$= (\alpha^{*2}_{1_2}, \alpha^{*2}_{2_2}, \alpha^{*2}_{3_2}).$$

Checking, we have

$$\alpha^{*2} Q_2^*$$

$$= \begin{pmatrix} .46784 & .32749 & .20468 \end{pmatrix} \begin{pmatrix} .3 & .7 & 0 \\ .5 & 0 & .5 \\ .8 & 0 & .2 \end{pmatrix}$$

$$= \begin{pmatrix} .46784 & .32749 & .20468 \end{pmatrix}.$$

5 Strategic Stability and Dynamic Consistency in Network Formation Games

Throughout this section, let $\sigma^*_D(\cdot) = (\sigma^*_d(\cdot|\cdot))_{d\in D}$ be a stationary equilibrium of the dynamic network formation game with corresponding equilibrium Markov transition

$$p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma^*_D(\cdot)),$$

unique basins of attraction,

$$\left\{A^{*1}, A^{*2}, \ldots, A^{*L}\right\},$$

and ergodic probability measures,

$$\mathcal{E}^* = \{\alpha^{*1}(\cdot), \ldots, \alpha^{*L}(\cdot)\},$$

where

$$\alpha^{*k}(\omega) = \frac{1}{E_\omega T^*_\omega} \text{ for all } \omega \in \Omega \text{ and } k = 1, \ldots, L,$$

and $E_\omega T^*_\omega$ is the expected hitting time (or expected number of moves) for the endogenous process $\{W^*_n\}_n$ governed by the equilibrium Markov transition $p^*(\cdot|\cdot)$ to reach network-coalition pair ω again after leaving ω at time zero.

5.1 Strategic Stability

Each player's equilibrium network proposal strategy

$$\omega = (G, S) \to \sigma^*_d(\cdot|G, S)$$

governs the way in which player d tries to influence the process of network formation across time and for each *given* status quo coalition S, $\sigma^*_d(\cdot|\cdot, S)$ is an equilibrium Markov transition on networks governing player d's network proposal process. For each status quo coalition S, we will refer to the equilibrium Markov transitions, $(\sigma^*_d(\cdot|\cdot, S))_{d\in D}$, as the *S-proposal transitions* and we will refer to the induced equilibrium Markov network-coalition transition, $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma^*_D(\cdot))$, as the *state transition.*

To begin, let $\mathcal{L}^*_{dS}$ denote the set of absorbing sets corresponding to player d's S-proposal transition $\sigma^*_d(\cdot|\cdot, S)$.[10] If the set of networks $\mathbb{E}$ is an absorbing set for player d under S-proposal transition $\sigma^*_d(\cdot|\cdot, S)$, then for any status quo network $G \in \mathbb{E}$, it is *optimal* for player $d \in S$ to propose with probability 1 either the status quo network or a new network G' in $\mathbb{E}$. Moreover, by assumption A-3(b) if $d \notin S$, then player d is

[10] A set of networks $\mathbb{E}$ is absorbing for player d in coalition S provided

$$\sigma^*_d(\mathbb{E}|G, S) = 1$$

for all $G \in \mathbb{E}$. The set $\mathbb{E}$ is minimal absorbing if there exists no *proper* subset of $\mathbb{E}$ which is also absorbing.

constrained to propose only the status quo network.[11] If in addition, $\mathbb{E}$ is absorbing for all players in S, that is, if $\mathbb{E} \in \cap_{d \in S} \mathcal{L}^*_{dS}$, then for all status quo networks $G \in \mathbb{E}$, it is optimal for *all* players to propose a network contained in $\mathbb{E}$ with probability 1. Note, however, that unless $\mathbb{E}$ is a singleton (i.e., $\mathbb{E} = \{G\}$ for some network $G \in \mathbb{G}$), players may not agree on their individual network proposals. However, if $\mathbb{E}$ is absorbing for all members of S, then at least all members will agree that their proposals should be drawn from $\mathbb{E}$. Thus, we can think of the sets in $\cap_{d \in S} \mathcal{L}^*_{dS}$ as being *strategically stable* for coalition S - as long as coalition S is the status quo coalition. We will denote by $\mathcal{L}^*_S$ the intersection $\cap_{d \in S} \mathcal{L}^*_{dS}$ and we will refer to the absorbing sets contained in $\mathcal{L}^*_S$ as S-strategically stable sets.

Let $\mathcal{C}$ be a subcollection of the feasible coalitions in $\mathcal{F}$. We will say that a set of networks $\mathbb{E}$ is $\mathcal{C}$-strategically stable if it is S-strategically stable for all coalitions $S \in \mathcal{C}$, that is, if

$$\mathbb{E} \in \cap_{S \in \mathcal{C}} \mathcal{L}^*_S := \mathcal{L}^*_{\mathcal{C}},$$

and we will say that $\mathbb{E}$ is strategically stable if $\mathcal{C} = \mathcal{F}$. Thus, if $\mathbb{E}$ is $\mathcal{C}$-strategically stable, then in any status quo state $\omega = (G, S)$ with $G \in \mathbb{E}$ and $S \in \mathcal{C}$, all players in S will find it in their best interest to propose networks in $\mathbb{E}$, while all players not in S will be constrained (under the rules of network formation) to propose the status quo network G - also a network in $\mathbb{E}$. Moreover, the same will be true in any other status quo state $\omega' = (G', S')$ with $G' \in \mathbb{E}$ and $S' \in \mathcal{C}$, that is, all players in S' will find it in their best interest to propose networks in $\mathbb{E}$, while all players not in S' will be constrained to propose the status quo network G'.

We have the following formal definition.

Definitions 7 (*$\mathcal{C}$-Strategic Stability and Strategic Stability*)

A set of networks $\mathbb{E}$ is $\mathcal{C}$-strategically stable for $\mathcal{C}$ a subcollection of feasible coalitions in $\mathcal{F}$, if in all states $(G, S) \in \mathbb{E} \times \mathcal{C}$ all players $d \in S$ propose networks in $\mathbb{E}$ with probability 1, that is, if for all $(G, S) \in \mathbb{E} \times \mathcal{C}$

$$\sigma^*_d(\mathbb{E}|G, S) = 1 \text{ for all } d \in S.$$

Thus, the set of all $\mathcal{C}$-strategically stable sets is given by

$$\mathcal{L}^*_{\mathcal{C}} := \cap_{S \in \mathcal{C}} \mathcal{L}^*_S := \cap_{S \in \mathcal{C}} \left[\cap_{d \in S} \mathcal{L}^*_{dS} \right].$$

If $\mathbb{E}$ is $\mathcal{F}$-strategically stable, then we say it is strategically stable. The collection of all strategically stable sets is given by

$$\mathcal{L}^*_{\mathcal{F}} := \cap_{S \in \mathcal{F}} \mathcal{L}^*_S := \cap_{S \in \mathcal{F}} \left[\cap_{d \in S} \mathcal{L}^*_{dS} \right].$$

[11] For any status quo state (G, S),

$$\sigma^*_d(\{G\}|G, S) = 1$$

for *any* player d not in coalition S. Thus, for any status quo state (G, S), $\{G\}$ is the minimal absorbing set for player outside the status quo coalition S.

Example 7: Strategic Stability in the Dynamic Club Network Formation Game

Returning to the club network formation game in Example 3, by examination of the pure stationary strategies in Table 3 we can conclude that the S-strategically stable sets (i.e., the coalitional stable sets) are given by

$$\mathcal{L}^*_{d_1\{d_1\}} := \{\{G_1, G_2, G_3\}, \{G_1, G_3\}, \{G_2\}, \{G_1\}\},$$
and
$$\mathcal{L}^*_{d_2\{d_2\}} := \{\{G_1, G_2, G_3\}, \{G_1, G_3\}, \{G_2, G_3\}, \{G_3\}\}.$$

It is even easier to conclude this after an examination of the S-proposal transition supernetworks depicted in Figure 6 corresponding to the proposal transitions $\sigma_{d_1}(\cdot|\cdot\{d_1\})$ and $\sigma_{d_2}(\cdot|\cdot\{d_2\})$.

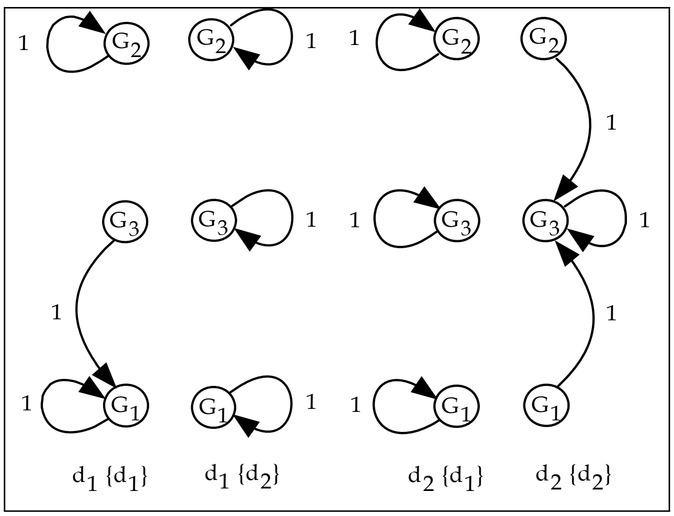

Figure 6: The Equilibrium Proposal Strategy Supernetworks

For example, in Figure 6 we see in the first column of figures, corresponding to player $d_1's$ proposal in state $(G_2, \{d_1\})$, that player d_1 chooses (as indicated in Table 3) proposal G_2 with probability 1.

Recalling that in Example 3 the feasible set of coalitions is given by $\mathcal{F}_1 = \{\{d_1\}, \{d_2\}\}$, the strategically stable sets are given by

$$\mathcal{L}^*_{\mathcal{F}_1} := \mathcal{L}^*_{d_1\{d_1\}} \cap \mathcal{L}^*_{d_2\{d_2\}} = \{\{G_1, G_2, G_3\}, \{G_1, G_3\}\}.$$

5.2 Dynamic Consistency

Suppose the $\mathcal{C}$-strategically stable set of networks $\mathbb{E}$ is such that *nature* chooses with probability 1 network-coalition pairs from $\mathbb{E}\times\mathcal{C}$ starting from any status quo network-coalition pair contained in $\mathbb{E}\times\mathcal{C}$. In particular, suppose that in addition to $\mathbb{E}$ being $\mathcal{C}$-strategically stable, $\mathbb{E}\times\mathcal{C}$ is absorbing for the state transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma^*_D(\cdot))$.

We will refer to any such set of networks as being $\mathcal{C}$-*dynamically consistent*. Thus, a set of networks $\mathbb{E} \in \mathcal{L}^*_{\mathcal{C}}$ is $\mathcal{C}$-dynamically consistent if $\mathbb{E} \times \mathcal{C} \in \mathcal{L}^*$, where $\mathcal{L}^*$ is the collection of all absorbing sets corresponding to the state transition $p^*(\cdot|\cdot)$.

We have the following formal definition.

Definitions 8 (*$\mathcal{C}$-Dynamic Consistency and Dynamic Consistency***)**

A $\mathcal{C}$-strategically stable set of networks $\mathbb{E}$ is $\mathcal{C}$-dynamically consistent if in all states $(G, S) \in \mathbb{E}\times\mathcal{C}$ nature chooses states in $\mathbb{E}\times\mathcal{C}$ with probability 1, that is,

$$p^*(\mathbb{E}\times\mathcal{C}|G, S) = 1 \textit{for all } (G, S) \in \mathbb{E}\times\mathcal{C}.$$

Thus, the set of all $\mathcal{C}$-dynamically consistent sets is given by

$$D^*_{\mathcal{C}} := \{\mathbb{E} \in \mathcal{L}^*_{\mathcal{C}} : \mathbb{E}\times\mathcal{C} \in \mathcal{L}^*\}.$$

If $\mathbb{E}$ is $\mathcal{F}$-dynamically consistent, then we say it is dynamically consistent. The collection of all dynamically consistent sets is given by

$$D^*_{\mathcal{F}} := \{\mathbb{E} \in \mathcal{L}^*_{\mathcal{F}} : \mathbb{E}\times\mathcal{F} \in \mathcal{L}^*\}.$$

Example 8: Dynamic Consistency in the Dynamic Club Network Formation Game

Returning again to the dynamic club network formation game in Example 3, by examination of the Figure 5 we can conclude that the collection of absorbing sets for the state transition $p^*(\cdot|\cdot)$ is given by

$$\mathcal{L}^* := \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_5, \omega_6\}\}.$$

From Example 5 we have $\mathcal{L}^*_{\mathcal{F}_1} = \{\{G_1, G_2, G_3\}, \{G_1, G_3\}\}$ and thus we have

$$\{G_1, G_2, G_3\} \times \mathcal{F}_1 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\},$$

and

$$\{G_1, G_3\} \times \mathcal{F}_1 = \{\omega_1, \omega_2, \omega_5, \omega_6\}.$$

We can conclude, therefore, that the set of strategically stable networks and the set dynamically consistent networks are equal (i.e., $\mathcal{L}^*_{\mathcal{F}_1} = \mathcal{D}^*_{\mathcal{F}_1}$).

The following result tells us precisely the relationship between dynamically consistent sets and basins of attraction.

Theorem 8 *(Dynamically Consistent Sets and Basins of Attraction)*

Let

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^\infty)_{d\in D} .$$

be a dynamic network formation game with state space $\Omega := \mathbb{G} \times \mathcal{F}$ *satisfying assumptions [A-1]-[A-4]. Also, let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^\infty$$

be the endogenous network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^*(\cdot))$ *where* $\sigma_D^*(\cdot)$ *is a stationary Nash equilibrium of* Γ.

If for some collection of player coalitions $\mathcal{C} \subseteq \mathcal{F}$, *the set of networks* $\mathbb{E}$ *is* $\mathcal{C}$-*dynamically consistent, then the set of network-coalition pairs* $E_{\mathcal{C}} := \mathbb{E}\times\mathcal{C}$ *contains at least one basin of attraction; that is,*

$$A^{*k} \subseteq E_{\mathcal{C}} \text{ for some } k = 1, \dots, L.$$

The conclusion of Theorem 8 is a consequence of two facts. First, if E is an absorbing set of the state transition $p^*(\cdot|\cdot)$, then $E \cap [\cup_k A^{*k}] \neq \emptyset$. If not then, recalling that $\Omega = \cup_k A^{*k} \cup T$, we must conclude that E is contained in the transient set T, a contradiction. Hence, it must be true that $E \cap A^{*k} \neq \emptyset$ for some k. But now because the endogenous process of network-coalition formation visits each network-coalition pair in A^{*k} infinitely often (i.e., because A^{*k} is recurrent) and because E is absorbing for $p^*(\cdot|\cdot)$, it must be true that $A^{*k} \subseteq E$ for all basins A^{*k} having a nonempty intersection with E. Thus, because $E_{\mathcal{C}} := \mathbb{E}\times\mathcal{C}$ is absorbing for $p^*(\cdot|\cdot)$, it must be true that $A^{*k} \subseteq E_{\mathcal{C}}$.

By Theorem 8, we conclude that starting at any network-coalition pair contained in $E_{\mathcal{C}}$, the network-coalition formation process $\{W_n^*\}_n$ will reach in finite time with probability 1 some basin of attraction A^{*k} contained in $E_{\mathcal{C}}$, and once there, will remain there.

5.3 Dynamic Path Dominance Core, Dynamic Strong Stability, and Dynamic Pairwise Stability

In a static abstract game setting Page and Wooders (2009) introduced the notion of the path dominance core. Stated informally, the path dominance core $\mathbb{C}$ contains all feasible networks G such that there is no domination path leading to another feasible network G' (i.e., $\mathbb{C}$ contains all undominated networks with respect to path dominance). A closely related notion, introduced earlier by Jackson and van den Nouweland (2005), also in a static setting, is the notion of strong stability. Stated loosely, a feasible network G is strongly stable if every player coalition that is able to change the status quo network G to another feasible network, prefers not to do so (i.e., stays with the status quo network). One way to extend the notions of the path dominance core and strongly stable networks to the dynamic setting considered here is as follows:

Definition 9 (*The Dynamic Path Dominance Core and Dynamically Strongly Stable Networks*)

A network $G^* \in \mathbb{G}$ is in the dynamic path dominance core (or equivalently, is dynamically strongly stable) if the set $\{G^*\}$ is dynamically consistent, that is, if $\{G^*\} \in \mathcal{L}^*_{\mathcal{F}}$ and $\{G^*\} \times \mathcal{F} \in \mathcal{L}^*$.

Thus, if a network is dynamically consistent, then all players want to stay there and the induced process abides by the wishes of the players.

The following result is a direct consequence of Theorem 9.

Theorem 9 *(The Dynamic Path Dominance Core, Dynamically Strongly Stable Networks, and Basins of Attraction)*

Let

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^\infty)_{d \in D} .$$

be a dynamic network formation game with state space $\Omega := \mathbb{G} \times \mathcal{F}$ satisfying assumptions [A-1]-[A-4]. Also, let

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^\infty$$

be the endogenous network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^*(\cdot))$ where $\sigma_D^*(\cdot)$ is a stationary Nash equilibrium of Γ.

If network $G^* \in \mathbb{G}$ is in the dynamic path dominance core, then the set of network-coalition pairs $\{G^*\} \times \mathcal{F}$ contains at least one basin of attraction; that is,

$$A^{*k} \subseteq \{G^*\} \times \mathcal{F} \text{ for some } k = 1, \ldots, L.$$

By Theorem 9, if network G^* is in the dynamic path dominance core, then there is a least one basin of attraction, say A^{*k}, consisting of network-coalition pairs of the form (G^*, S) for $S \in \mathcal{C}^{*k} \subseteq \mathcal{F}$ for some subcollection of player coalitions. Thus, starting at *any* network-coalition pair (G^*, S) (i.e., where S is any feasible coalition), the network-coalition formation process $\{W_n^*\}_n$ will reach in finite time with probability 1 a basin of attraction A^{*k} of the form $\{G^*\} \times \mathcal{C}^{*k}$ and once there will remain there.

Note that if for some network $G^* \in \mathbb{G}$ and some coalition $S^* \in \mathcal{F}$, $\{G^*\} \in \mathcal{L}^*_{S^*}$ and $\{(G^*, S^*)\} \in \mathcal{L}^*$, so that $\{G^*\}$ is $\{S^*\}$-dynamically consistent, this does not necessarily imply that G^* is in the dynamic path dominance core, even if $\{(G^*, S^*)\}$ basin of attraction, because $\{G^*\}$ may not be dynamically consistent. Why? Because while nature will choose with probability 1 the network-coalition pair (G^*, S^*) if the status quo is (G^*, S^*), if the status quo coalition is not S^*, that is, if the status quo state is (G^*, S') for some coalition $S' \in \mathcal{F}$ not equal to S^*, some players in S' may propose a network other than G^* (i.e., it may be the case that $G^* \notin \mathcal{L}^*_{dS'}$ for some player $d \in S'$) and in turn nature may choose a state other than (G^*, S^*). Moreover, if G^* is not strategically stable, but nonetheless $\{G^*\} \times \mathcal{C} \in \mathcal{L}^*$ for some subset of coalitions $\mathcal{C} \subseteq \mathcal{F}$, then if the equilibrium network-coalition formation process reaches any state $(G^*, S) \in \{G^*\} \times \mathcal{C}$, the process will remain in the set $\{G^*\} \times \mathcal{C}$ - despite network proposals to the contrary by players, even players in coalitions in $\mathcal{C}$. In such a case, the state transition overrides the wishes of the players. This leads to the following alternative notion of dynamic path dominance core.

Definition 9' (*The State Transition Core*)

(1) (State Transition Core) A network $G^* \in \mathbb{G}$ is in the state transition core if the set of states $\{G^*\} \times \mathcal{F}$ is an absorbing set for the state transition $p^*(\cdot|\cdot)$.

(2) (Weak State Transition Core) A network $G^* \in \mathbb{G}$ is in the weak state transition core if the set of states $\{G^*\} \times \mathcal{C}$ is an absorbing set for the state transition $p^*(\cdot|\cdot)$ for some subset of coalitions $\mathcal{C} \subseteq \mathcal{F}$.

Under the definition of weak state transition core, for any basin of attraction A^{*k} of the form $A^{*k} = \{(G^{*k}, S^{*k})\}$, G^{*k} is in the weak state transition core. Moreover, if for some state transition absorbing set E, E contains $A^{*k} = \{(G^{*k}, S^{*k})\}$ and is disjoint from the other basins, then starting at any network-coalition pair in E, the process will reach in finite time with probability 1 the network-coalition pair (G^{*k}, S^{*k}) and will remain there.

To extend the definition of the pairwise stability introduced in Jackson and Wolinsky (1996) to the dynamic setting considered here, we begin by specializing the feasible set of coalitions to coalitions of size no greater than 2.[12]

Definition 10 (*Dynamic Pairwise Stability*)

Suppose the feasible set of coalitions is given by

$$\mathcal{F}_2 = \{S \in P(D) : |S| \leq 2\}.$$

(i.e., all feasible coalitions consist of at most two players). Then a network $G^* \in \mathbb{G}$ is dynamically pairwise stable if the set $\{G^*\}$ is dynamically consistent, that is, if $\{G^*\} \in \mathcal{L}^*_{\mathcal{F}_2}$ and $\{G^*\} \times \mathcal{F}_2 \in \mathcal{L}^*$.

The following result is also a direct consequence of Theorem 9 and the definition of dynamic pairwise stability.

Theorem 10 (*Dynamic Pairwise Stability and Basins of Attraction*)

Let

$$\Gamma := (\Omega, E_d(\cdot)(\cdot), \Pi_d^{\infty})_{d \in D}.$$

[12] Stated very loosely, a network G is pairwise stable if every 1 or 2 player coalition that is able to change the status quo network G to another network, prefers not to (i.e., stays with the status quo network).

be a dynamic network formation game with state space $\Omega := \mathbb{G} \times \mathcal{F}$ *satisfying assumptions [A-1]-[A-4]. Also, let*

$$\{W_n^*\}_n = \{(G_n^*, S_n^*)\}_{n=1}^{\infty}$$

be the endogenous network-coalition formation process governed by the equilibrium Markov transition $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma_D^*(\cdot))$ *where* $\sigma_D^*(\cdot)$ *is a stationary Nash equilibrium of* Γ.

If network $G^* \in \mathbb{G}$ *is dynamically pairwise stable, then the set of network-coalition pairs* $E_{\mathcal{F}_2} := \{G^*\} \times \mathcal{F}_2$ *contains at least one basin of attraction; that is,*

$$A^{*k} \subseteq E_{\mathcal{F}_2} \text{ for some } k = 1, \ldots, L.$$

By Theorem 10, if network G^* is dynamically pairwise stable, then there is a least one basin of attraction, say A^{*k}, consisting of network-coalition pairs of the form (G^*, S) for $S \in \mathcal{C}_2^{*k} \subseteq \mathcal{F}_2$. Thus, starting at *any* network-coalition pair (G^*, S), the network-coalition formation process $\{W_n^*\}_n$ will reach in finite time with probability 1 a basin of attraction A^{*k} of the form $\{G^*\} \times \mathcal{C}_2^{*k}$ and once there will remain there. Moreover, the ergodic measure $\alpha^{*k}(\cdot)$ is such that $\alpha^{*k}(\{G^*\} \times \mathcal{C}_2^{*k}) = 1$. Thus, by Theorem 10, for any dynamically pairwise stable network G^*, any state (G^*, S) with player coalition $S \in \mathcal{C}_2^{*k}$ is contained in the support of the unique ergodic measure $\alpha^{*k}(\cdot)$ corresponding to the basin of attraction $\{G^*\} \times \mathcal{C}_2^{*k}$. This conclusion is similar to the conclusion reached by Jackson and Watts (2002) for pairwise stable networks resulting from a network formation process governed by a Markov chain generated by myopic players. They reach their conclusion by considering a sequence of perturbed irreducible Markov chains (i.e., each with a unique invariant measure) converging to the original Markov chain. This method is similar to a method introduced into games theory by Young (1993) which in turn is based on some very general perturbation methods found in Freidlin and Wentzell (1984). Here we have reached a similar conclusions in a different but related model using a different approach.

Example 6: Emptiness of the Dynamic Path Dominance Core

Returning again to the club network formation game in Example 3, we see by a re-examination of Figure 5, reproduced in Figure 7 below, that for the club network formation game in Example 3 the path dominance core is empty.

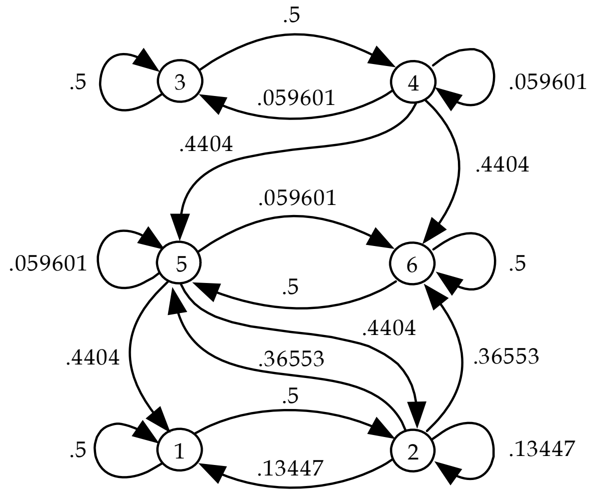

Figure 7: The Equilibrium Markov Supernetwork from Example 3

Again note, however, that the set of networks $\{G_1, G_3\}$ is dynamically consistent and that the set of network-coalition pairs

$$A^* = \{\omega_1, \omega_2, \omega_5, \omega_6\}$$

is the only basin of attraction.

References

[1] Bala, V. and Goyal, S. (2000) A Noncooperative Model of Network Formation," *Econometrica* 68, 1181-1229.

[2] Blackwell, D. (1965) "Discounted Dynamic Programming," *Annals of Mathematical Statistics* 36, 226-235.

[3] Chwe M. (1994) "Farsighted Coalitional Stability," *Journal of Economic Theory* 63, 299-325.

[4] Durrett, R. (2005) *Probability: Theory and Examples* (3rd edition), Brooks/Cole, Australia.

[5] Federgruen, A. (1978) "On N-Person Stochastic Games with Denumerable State Space," *Advances in Applied Probability* 10, 452-471.

[6] Filar, J. A., Schultz, T. A., Thuijsmann, F., and Vrieze, O. J. (1991) "Nonlinear Programming and Stationary Equilibria in Stochastic Games," *Mathematical Programming: Series A and B* 50, 227-237.

[7] Freidlin, M., and Wentzell, A. (1984) *Random Perturbations of Dynamical Systems*, pp 161-211, Springer-Verlag, New York.

[8] Herings, P. J.-J., and Peeters, Ronald, J. A. P. (2004) "Stationary Equilibria in Stochastic Games: Structure, Selection, and Computation," *Journal of Economic Theory* 118, 32-60.

[9] Jackson, M. O. and van den Nouweland, A. (2005) "Strongly Stable Networks," *Games and Economic Behavior* 51, 420-444.

[10] Jackson, M. O. and A. Wolinsky (1996) "A Strategic Model of Social and Economic Networks," *Journal of Economic Theory* 71, 44-74.

[11] Jackson, M. and Watts, A. (2002) "The Evolution of Social and Economic Networks," *Journal of Economic Theory* 106, 265-295.

[12] Kemeny, J. G., Snell, J. L. Knapp, A. W. (1976) *Denumerable Markov Chains*, Springer-Verlag, New York.

[13] Norris, J. R. (1997) *Markov Chains*, Cambridge University Press, Cambridge.

[14] Page, F. H., Jr. and Wooders, M. (2007) "Networks and Clubs," *Journal of Economic Behavior and Organization* 64, 406-425.

[15] Page, F. H., Jr. and Wooders, M. (2009) "Strategic Basins of Attraction, the Path Dominance Core, and Network Formation Games," *Games and Economic Behavior* 66, 4562-487

[16] Page, F. H., Jr., Wooders, M., Kamat, S. (2005), "Networks and Farsighted Stability," *Journal of Economic Theory* 120, 257-269.

[17] Skyrms, B. and Pemantle, R. (2000) "A Dynamic Model of Social Network Formation," Proceedings of the National Academy of Sciences 97, 9340-9346.

[18] Strook, D. W. (2005) *An Introduction to Markov Processes*, Springer-Verlag, Berlin.

[19] Watts, A. (2001) "A Dynamic Model of Network Formation," *Games and Economic Behavior* 34, 331-341.

[20] Young, H. P. (1993) "The Evolution of Conventions," *Econometrica* 61, 57-84.